Assessment Three

Integral Calculus for Life Sciences

UCR Math-007B-B01, Summer 2019

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This assessment is intended not only as a way to determine if you've understood what you were asked to think about for the homework, but also as a means for you to assess your own understanding of that material, and assess if you're meeting your own expectations for yourself. I expect that it'll be challenging for anyone to respond to all these prompts in the allotted time, but that's okay. It wouldn't be a very useful assessment if it wasn't adequately challenging, and furthermore you'd be surprised how much you actually *learn* when you're challenged and under a bit of pressure.

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1. The function $P'(t)$ models the rate of growth of the US population in the year t. Explain what this definite integral represents.

$$\int_{1992}^{2019} P'(t)\,\mathrm{d}t$$

The area $\int_{1992}^{2019} P'(t)\,dt$ corresponds to the "amount of population accumulated" ~~between~~ from 1992 to 2019. I.e. it's how much the population has increased by since 1992 in the US.

2. A particle moves along the x-axis with velocity given by by the function

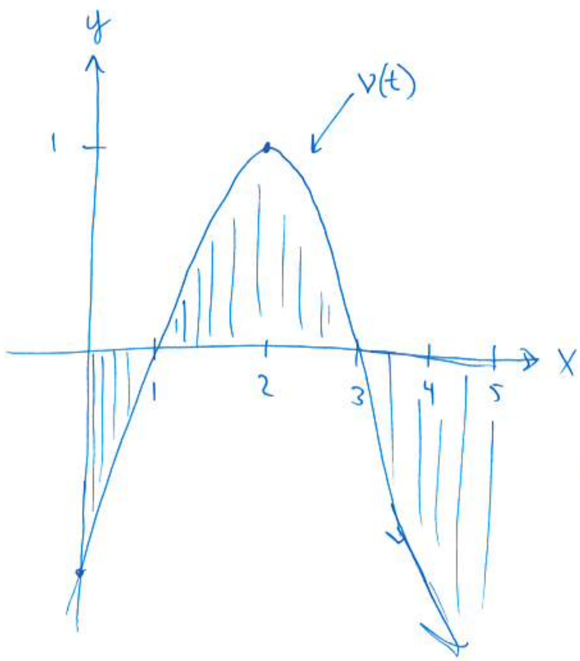
$$v(t) = -(t-2)^2 + 1$$

for $t \in [0,5]$. Declare the location of the particle at time zero to be the origin.

(a) Graph the function $v(t)$, and use the graph to determine when the particle moves to the right and when the particle moves to the left.

(b) Write a function for the location of the particle for times $t \in [0,5]$.

(c) Find the left-most and right-most positions on the axis that the particle reaches.



(a) Recall the moves to the right when its velocity is positive, and to the left when its velocity is negative, so its moving to the right on $(1,3)$ and to the left on $(0,1) \cup (3,5)$.

(b) The location function will be an antiderivative of the velocity function

$$p(t) = \int v(t)\,dt = \int -t^2 + 4t - 3\,dt$$

$$= -\frac{1}{3}t^3 + 2t^2 - 3t + C$$

(c) These will correspond to max and min values of $p(t)$, which occur either at the boundary of the domain, or when $p'(t) = v(t) = 0$. So our four candidate times are $t = 0, 1, 3, 5$. Trying each,

$p(0) = C$

$p(1) = -\frac{1}{3} + 2 - 3 + C = \frac{3C-4}{3} = C - \frac{4}{3}$

$P(3) = C$

$P(5) = -\frac{125}{3} + 50 - 15 + C \approx C - \frac{20}{3}$

Furthest right position is C, and furthest left position is $C - \frac{20}{3}$.

3. For a fish that starts life with a length of 1 cm and has a maximum length of 30 cm, the von Bertalanffy growth model predicts that the growth rate of the fish fish is $29e^{-a}$ cm/year, where a is the age of the fish. What is the average length of the fish over its first five years of life?

$$\ell'(a) = 29e^{-a}$$

$$\Rightarrow \ell(a) = \int \ell'(a)\, da = -29e^{-a} + C$$

Since $\ell(0) = 1$, $C = 30$, so $\ell(a) = -29e^{-a} + 30$.

Then the average will be

$$\frac{1}{5-0}\int_0^5 -29e^{-a} + 30\, da = \frac{1}{5}\left(29e^{-a} + 30a\right)\Big|_0^5$$

$$= \frac{29}{5e^5} + 30 - \frac{29}{5}$$

4. Suppose the volume of a cell is increasing at a constant rate of $10^{-6}\,\text{cm}^3/\text{s}$. Write down a function $V(t)$ that models the volume of the cell at time t, given that the volume of the cell at time $t = 0$ seconds is 13×10^{-5}. What is the volume of the cell at $t = 10$ seconds?

$$V'(t) = 10^{-6} \Rightarrow V(t) = 10^{-6}t + C$$

$$\Rightarrow V(t) = 10^{-6}t + 13 \times 10^{-5}$$

since $V(0) = 13 \times 10^{-5}$

at $t = 10$ seconds

$$V(10) = 10^{-6}(10) + 13 \times 10^{-5}$$

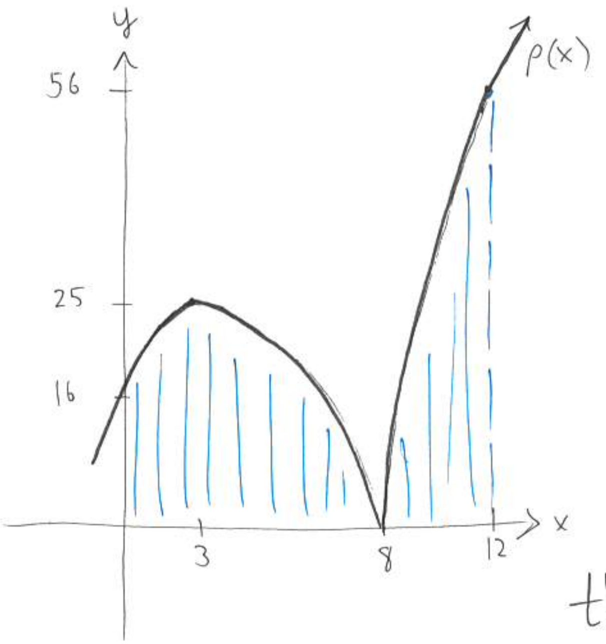
$$= 10^{-5} + 13 \times 10^{-5}$$

$$= 14 \times 10^{-5}$$

5. You're studying a population of trout in a 12 mile section of stream. The population density of trout in this section of stream is given by the function

$$\rho(x) = |-x^2 + 6x + 16| = \left|-(x-8)(x+2)\right|$$

where $\rho(x)$ is measured in trout per mile and x is measured in miles running from 0 to 12 miles. Graph the function ρ and use the graph to find the places along the stream where the density of trout is minimal and maximal. What is the total number of trout in the stream? What is the average density of trout along the 12 mile stretch of stream? Based on your graph, find a location along the stream where the trout population density is about the same as the *average* trout population density along the entire 12 mile stretch of stream.



The density is minimal at $x=8$ and maximal at the end of the stream where $x=12$.

The total number of trout in this stream will be given by the area under the curve of the graph of $\rho(x)$. So

$$\int_0^8 -x^2+6x+16\,dx + \int_8^{12} x^2-6x-16\,dx = \left(-\tfrac{1}{3}(8)^3+3(8)^2+16(8)\right)2 + \left(\tfrac{1}{3}(8)^3-3(8)^2-16(8)\right)$$

$$= \frac{8^2}{3}\left(\left(-8+9+6\right)2+\left(8-9-6\right)\right)$$

$$= \frac{64}{3}(7) \approx 149 \text{ trout}$$

So the average density is $\frac{149}{12} \approx 12$

It looks like this average occurs at about $x=7$ and $x=8+\frac{1}{2}$

6. Calculate the arclength of the curve in the (x,y)-plane given by $y = \left(\frac{2}{3}t - 1\right)^{3/2}$ between the two points $\left(4, \sqrt{\frac{625}{27}}\right)$ and $(9, \sqrt{625})$.

$$y' = \frac{3}{2}\left(\frac{2}{3}t-1\right)^{1/2}\left(\frac{2}{3}\right)$$

$$\int_4^9 \sqrt{1+(y'(t))^2}\,dt = \int_4^9 \sqrt{1+\left(\frac{2}{3}t-1\right)}\,dt = \ldots$$

$$\ldots = \frac{\sqrt{2}}{\sqrt{3}}\int_4^9 t^{1/2}\,dt = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{2}{3}t^{3/2}\Big|_4^9\right) = \frac{2\sqrt{2}}{3\sqrt{3}}\left(9^{3/2} - 4^{3/2}\right)$$

$$= \frac{2\sqrt{2}\sqrt{3}}{3\cdot 3}\left(3^3 - 2^3\right) = \frac{38\sqrt{6}}{3}$$

7. Suppose you have two poles that are posted at $x = -M$ and $x = M$, and between the two poles you hang a cable. The cable will droop under the force of gravity, with it's lowest point being over $x = 0$. The shape that this cable makes is called a catenary, and is modelled by the graph of the equation

$$y = \frac{1}{2a}\left(e^{ax} + e^{-ax}\right),$$

where the constant a depends on any slack is in the cable and the force of gravity. Compute the length of such a cable if $a = 1$ and $M = \ln(2)$.

$$\int_{-\ln(2)}^{\ln(2)} \sqrt{1+(y'(x))^2}\,dx = \int_{-\ln(2)}^{\ln(2)} \sqrt{1+\frac{1}{4}\left(e^{2x}+e^{-2x}-2e^{2x-2x}\right)}\,dx$$

the integrand is odd b/c it models a caternary.

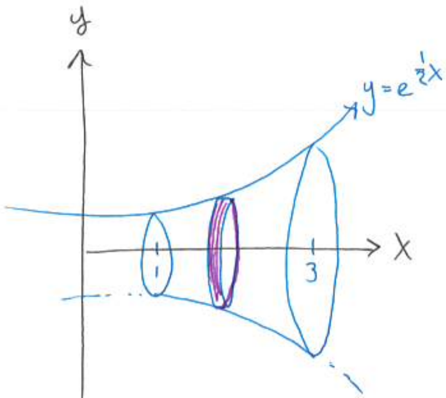
$$= \int_0^{\ln(2)} \sqrt{\frac{e^{2x}}{4}+\frac{1}{2}+\frac{e^{-2x}}{4}}\,dx = \int_0^{\ln(2)} \sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\int_0^{\ln(2)} e^x + e^{-x}\,dx = \frac{1}{2}\left(e^x - e^{-x}\right)\Big|_0^{\ln(2)}$$

$$= \frac{1}{2}\left(e^{\ln(2)} - e^{-\ln(2)}\right) - \frac{1}{2}\left(e^0 - e^{-0}\right)$$

$$= \frac{1}{2}\left(2 - \frac{1}{2}\right) - 0 = \frac{3}{4}$$

7. Calculate the volume of the solid formed by rotating the region bounded between the curves $y = e^{\frac{1}{2}x}$, $y = 0$, $x = 1$, and $x = 3$, about the x-axis.

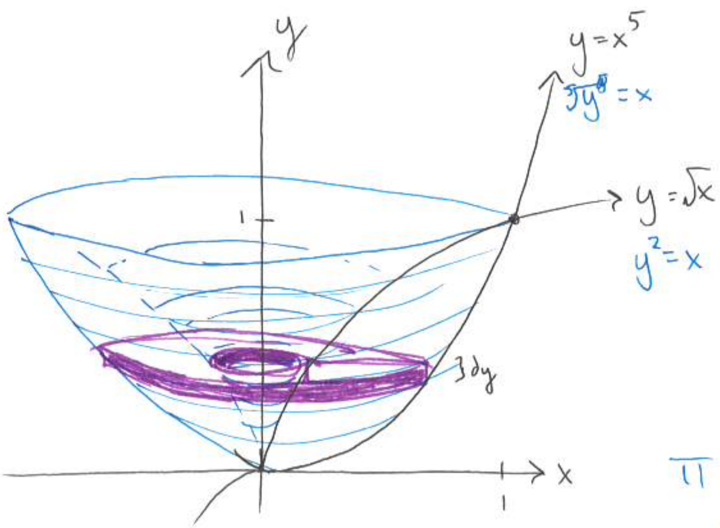


$$\int_1^3 \pi \left(e^{\frac{1}{2}x}\right)^2 dx$$

$$= \pi \int_1^3 e^x \, dx = \pi e^x \Big|_1^3$$

$$= \pi \left(e^3 - e\right)$$

8. Calculate the volume of the solid formed by rotating the region bounded between the curves $y = \sqrt{x}$ and $y = x^5$ about the y-axis.



$$\int_0^1 \pi R^2 - \pi r^2 \, dy$$

when $R = \sqrt[5]{y}$ and $r = y^2$

$$\pi \int_0^1 \left(\sqrt[5]{y}\right)^2 - \left(y^2\right)^2 dy$$

$$= \pi \left(\frac{5}{7} y^{7/5} - \frac{1}{5} y^5\right)\Big|_0^1 = \pi \left(\frac{5}{7} - \frac{1}{5}\right) = \frac{18\pi}{35}$$

10. (RECREATIONAL) Three friends Anita, Becca, and Charleston are challenged to a game by the Game Maestro. The Game Maestro places two colored dots on each of the friends' foreheads and tells the friends that each dot is either blue or yellow, but neither color is used more than four times. He then places the three friends in a circle so that each of them can see the dots on their friends' foreheads, but not on their own. Then the game goes like this: The Maestro will ask the friends in turn, first Anita, then Becca, then Charleston, then Anita again, then Becca again, and so on, if they know the colors of the dots on their foreheads. If someone responds "no," the Maestro asks the next person. If someone responds "yes" and is right, the friends win! Whereas if someone responds "yes" and is wrong, all three friends will be banished to the shadow realm.

The friends were given no time to strategize, but they begin playing. Their responses in turn are

no no no no yes

and the three friends win! Whare are the colors of the dots on Becca's forehead?

See Mike :)