How to integrate a rational function $\frac{P(x)}{Q(x)}$.

1. If $\deg P \geq \deg Q$ the perform polynomial long division (or another procedure) to write $\frac{P(x)}{Q(x)}$ as (a polynomial) $+ \frac{R(x)}{Q(x)}$ where $\deg R < \deg Q$. Then integrating (a polymiobnic) is chill and you can continue working on $\frac{R(x)}{Q(x)}$.

2. Factor $Q(x)$ into irreducible linear and quadratic factors, and if any of these factors cancel with the numerator, do so. Actually factoring $Q(x)$ may be really hard, so in exercises it'll likely be (nearly) factored for you.

*3. Write down the Partial Fraction Decomposition based on the factors of $Q(x)$. This is the new/involved step.

4. Break up the integral over the Partial Fraction Decomposition and integrate each summand individually. The summands should look something like either

$$\frac{A}{X+B} \quad \text{OR} \quad \frac{Ax+B}{X^2+Cx+D},$$

which wân seen how to deal with before. Remember

$$\int \frac{A}{X+B}\,dx = A\ln|X+B| + C \quad \text{and} \quad \int \frac{A}{X^2+B}\,dx = \frac{A}{B}\arctan\left(\frac{X}{B}\right) + C.$$

* The partial fraction decomposition of a rational expression.

Rather than explain how to do this in general, its easier to see the pattern through cases/examples/templates.

- If the denominator has <u>distinct</u> <u>linear</u> factors ...

$$\frac{P(x)}{(x-a)(x-b)(x-c)(x-d)} \overset{\text{dream}}{=} \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d}$$

- If the denominator has <u>duplicate</u> <u>linear</u> factors ...

$$\frac{P(x)}{(x-a)^5} \overset{\text{dream}}{=} \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-a)^4} + \frac{E}{(x-a)^5}$$

- If the denominator has <u>distinct</u> <u>quadratic</u> factors ...

$$\frac{P(x)}{(x^2+ax+b)(x^2+cx+d)(x^2+ex+f)} \overset{\text{dream}}{=} \frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d} + \frac{Ex+F}{x^2+ex+f}$$

- If the denominator has <u>duplicate</u> <u>quadratic</u> factors ...

$$\frac{P(x)}{(x^2+ax+b)^3} \overset{\text{dream}}{=} \frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{(x^2+ax+b)^2} + \frac{Ex+F}{(x^2+ax+b)^3}$$

Then the general case will be some combination of these four scenarios.