Math135 Engineering Calculus I

Yet Another Pop Quiz

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NAME: ______________________________

1. Consider the function f defined by the formula $f(x) = x\sec(x)$, and consider the graph $y = f(x)$. Determine an equation for the line tangent to this graph at that point where $x = \frac{\pi}{3}$. Express the equation in the form $y = mx + b$.

   Since $f\left(\frac{\pi}{3}\right) = \frac{2\pi}{3}$, the coordinates of the point of tangency are $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$. And since

   $$f(x) = x\sec(x) \qquad \Longrightarrow \qquad f'(x) = x\sec(x)\tan(x) + \sec(x)\,,$$

   the slope of the tangent line at this point will be $f'\left(\frac{\pi}{3}\right) = \frac{2\sqrt{3}\pi}{3} + 2$. Altogether, an equation of the tangent line is $\left(y - \frac{2\pi}{3}\right) = \left(\frac{2\sqrt{3}\pi}{3} + 2\right)\left(x - \frac{\pi}{3}\right)$, or

   $$y = \left(\frac{2\sqrt{3}\pi}{3} + 2\right)x - \frac{2\sqrt{3}\pi^2}{9}\,.$$

2. Consider the curve in the xy-plane that consists of all the points (x, y) that satisfy the equation $x^2y^3 + 3 = 7y^2$. Note that this curve passes though the point $(2, 1)$. Determine an equation for the line tangent to this curve at that point. Express the equation in the form $y = mx + b$.

   Since $(2)^2(1)^3 + 3 = 7(1)^2$, the point $(2, 1)$ does in fact lie on the curve. Implicitly differentiating the equation for the curve,

   $$x^2y^3 + 3 = 7y^2 \quad \Longrightarrow \quad 2xy^3 + x^2\left(3y^2y'\right) = 14yy' \quad \Longrightarrow \quad y' = \frac{2xy^2}{14 - 3x^2y}\,.$$

   Evaluating this expression for y' at $x = 2$ and $y = 1$, will give us the slope of the tangent line at this point.

   $$y'\Big|_{\substack{x=2\\y=1}} = \frac{2(2)(1)^2}{14 - 3(2)^2(1)} = 2$$

   So an equation of the tangent line is $y - 1 = 2(x - 2)$, or $y = 2x - 3$.

3. Write down a concise closed-form formula for the derivative of each of the following functions.

$$f(x) = x^5 + \frac{1}{x^5} + \frac{x}{5} + \frac{5}{x} + \sqrt[5]{x} + 5$$

$$f'(x) \;=\; 5x^4 - \frac{5}{x^6} + \frac{1}{5} - \frac{5}{x^2} + \frac{1}{5x^{4/5}}.$$

$$h(t) = \tan\big(\cos(t)\big)$$

Per the chain rule, $h'(t) \;=\; -\sec^2\big(\cos(t)\big)\sin(t)$.

$$g(x) = x^8\left(x^3-1\right)^3$$

$$\begin{aligned} g'(x) \;&=\; x^8\Big(3\left(x^3-1\right)^2\Big)\left(3x^2\right) + 8x^7\left(x^3-1\right)^3 \;=\; 9x^{10}\left(x^3-1\right)^2 + 8x^7\left(x^3-1\right)^3 \\ &=\; x^7\left(x^3-1\right)^2\Big(17x^3-8\Big). \end{aligned}$$

$$j(\theta) = \frac{2\theta - 3}{\theta\sin(8\theta)}$$

Per the quotient rule and product rule and chain rule,

$$j'(\theta) \;=\; \frac{2\Big(\theta\sin(8\theta)\Big) - (2\theta-3)\Big(8\theta\cos(8\theta) + \sin(8\theta)\Big)}{\Big(\theta\sin(8\theta)\Big)^2} \;=\; \frac{8x(3-2\theta)\cos(8\theta) + 3\sin(8\theta)}{\theta^2\sin^2(8\theta)}.$$

4. What is the exact values of this limit? If the value of the limit is not defined (does not exist), indicate this by simply crossing the expression out.

$$\lim_{h\to 0} \frac{\sqrt[3]{4+h}-\sqrt[3]{4}}{h}$$

This limit *looks like* the definition of the derivative of a function Upon careful inspection it's the derivative of the function $f(x) = \sqrt[3]{x}$ evaluated at $x = 4$. Since $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt[3]{x}\right) = \frac{1}{3}x^{-2/3}$,

$$\lim_{h\to 0} \frac{\sqrt[3]{4+h}-\sqrt[3]{4}}{h} = \frac{1}{3}x^{-2/3}\Big|_{x=4} = \frac{1}{3}(4)^{-2/3} = \frac{1}{6\sqrt[3]{2}}.$$

5. A drag race, commonly called a *pass*, is run on a 1320 ft (¼ mile) long track. Suppose $f(t) = t^3 + 32t$ serves as a model for how far a certain dragster travelled from the starting line t seconds after it began its pass until $t = 10$ seconds when it crossed the finish line.

(a) How fast was the dragster going at exactly the seven second mark?

Since $f'(t) = 3t^2 + 32$, the dragster was going $f'(7) = 3(7)^2 + 32 = 179$ ft/s at the seven second mark.

(b) What was the dragster's velocity the moment it crossed the finish line?

Since $f'(t) = 3t^2 + 32$, the dragster's velocity was $f'(10) = 3(10)^2 + 32 = 332$ ft/s at the ten second mark when it crossed the finish line.

(c) What was the dragster's *acceleration* the moment it crossed the finish line?

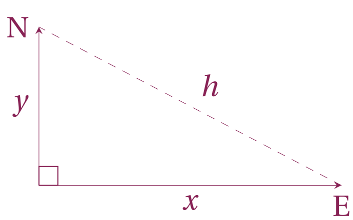
Since $f''(t) = 6t$, the dragster's acceleration was $f''(10) = 6(10) = 60$ ft/s^2 at the ten second mark when it crossed the finish line.

6. A Toyota Corolla and Honda Accord are sitting next to each other on a salt flat in the Utah desert, the Toyota pointing due north and the Honda pointing due east, the corners of their back bumpers touching. At the same moment, they each take off in the direction they're pointed at a constant speed, the Toyota at 95 ft/s and the Honda car at 168 ft/s.

   (a) At what rate is the distance between the cars increasing one minute after they take off?

   (b) Suppose that instead of taking off from the same spot, the Honda got a ¼ mile head start eastward. Now at what rate is the distance between the cars increasing one minute after they take off?

   (c) Suppose that the cars start bumper-to-bumper, but instead of pointing in orthogonal directions the Honda is pointing N30°E — i.e. its and the Toyota's trajectories differ by 30°. Now at what rate is the distance between the cars increasing one minute after they take off?



Let y denote the Toyota's distance from the starting spot and let x denote the Honda's distance from that same spot. We are given that $\dot{y} = 95$ ft/s and $\dot{x} = 168$ ft/s. The distance h between the two cars is related to y and x by the equation $h^2 = y^2 + x^2$. At the one minute (sixty second) mark, $y = 5700$ ft and $x = 10080$ ft, and so $h = 11580$ ft.

Differentiating the equation relating y and x and h we get $2h\dot{h} = 2y\dot{y} + 2x\dot{x}$, from which we can solve for $\dot{h}$ at the one minute mark:

$$\dot{h} \;=\; \frac{y\dot{y} + x\dot{x}}{h} \;=\; \frac{(5700)(95) + (10080)(168)}{(11580)} \;\approx\; 193 \text{ ft/s}$$

If the Honda were to have a ¼ mile (1320 ft) head start, its distance from the Toyota's starting location would be $x = 1320 + 10080 = 11400$ ft, and the equation relating y and x and h would instead be $h^2 = (1320 + x)^2 + y^2$. Again, differentiating this equation and solving for $\dot{h}$:

$$\dot{h} \;=\; \frac{y\dot{y} + (1320 + x)\dot{x}}{h} \;=\; \frac{(5700)(95) + (11400)(168)}{(11580)} \;\approx\; 212.15026 \text{ ft/s}$$

If the Honda's trajectory were instead N30°E, the equation relating y and x and h would be $h^2 = y^2 + x^2 - \sqrt{3}yx$ (law of cosines). Again, differentiating this equation and solving for $\dot{h}$:

$$\dot{h} \;=\; \frac{y\dot{y} + x\dot{x} - \frac{\sqrt{3}}{2}\big(y\dot{x} + \dot{y}x\big)}{h} \;=\; \frac{(5700)(95) + (10080)(168) - \frac{\sqrt{3}}{2}\big((5700)(168) + (95)(10080)\big)}{(11580)}$$

$$\approx\; 49.76927 \text{ ft/s}$$

7. Imagine a planet covered entirely in water with the exception of a single tiny island. On this island is a large fleet of airplanes and a bottomless supply of fuel. These airplanes have two peculiar features: on a single full tank of fuel each plane can each fly exactly half-way around their planet, and each plane is capable of exchanging fuel with another plane instantaneously midair. What is the fewest number of planes that would be needed to allow a single plane to fly completely around the planet along a great circle, such that every plane makes it safely back to the island?