Math135 Engineering Calculus I

This Quiz be Poppin'

Colorado Mesa University

NAME: ______________________________

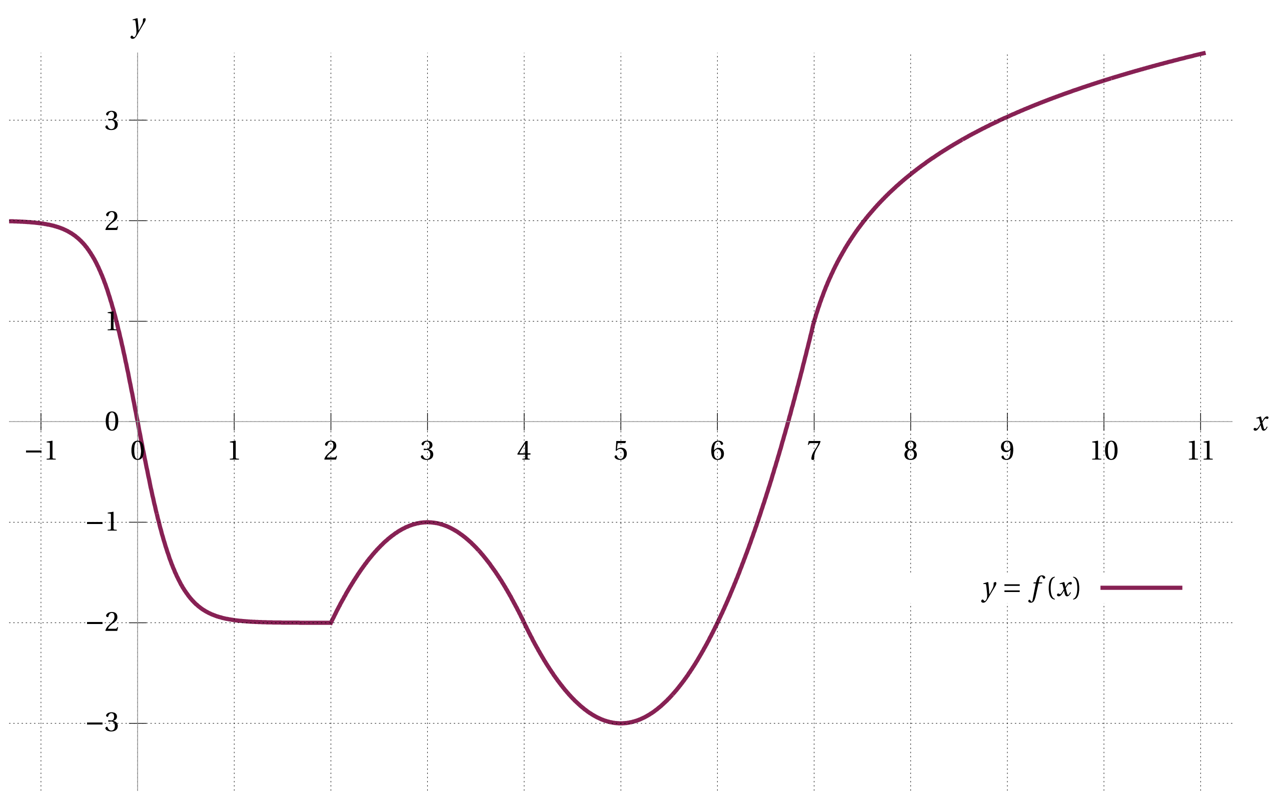
1. Given a continuous differentiable function f such that $f(0) = 27$ and $-\frac{1}{2} \le f'(x) \le 2$ for all real numbers x, cross out every number that *cannot* possibly be the value of of $f(36)$.

   $\cancel{-40}$ $\cancel{-30}$ $\cancel{-20}$ $\cancel{-10}$ $\cancel{0}$ 10 20 30 40 50 60 70 80 90 $\cancel{100}$ $\cancel{110}$ $\cancel{120}$ $\cancel{130}$ $\cancel{140}$

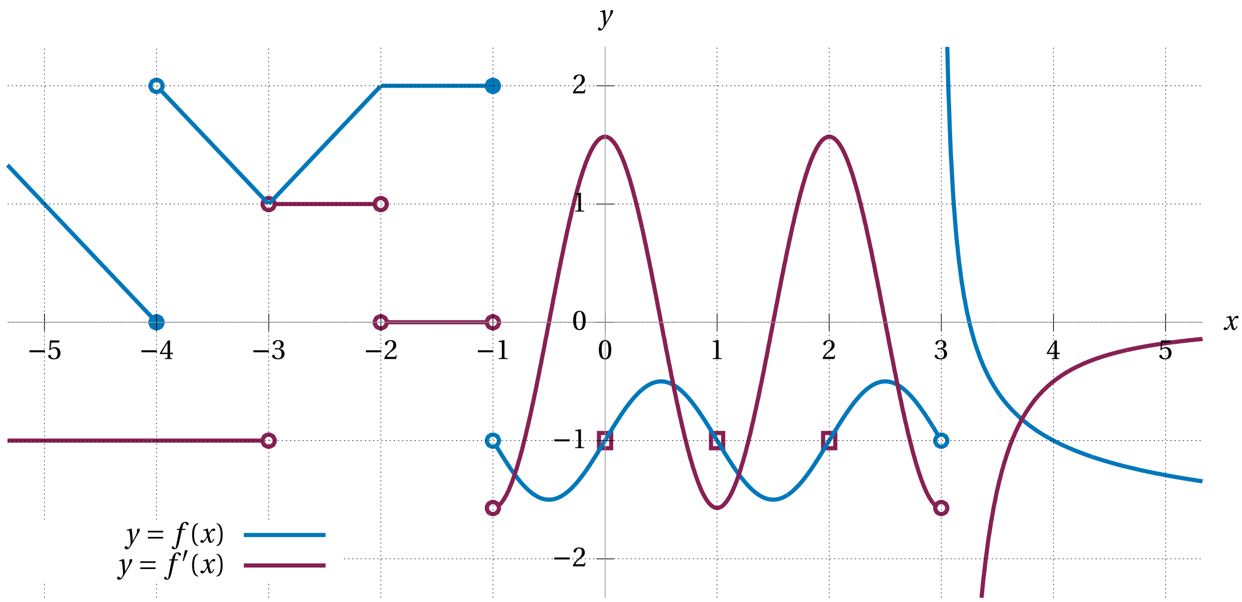
   The condition $-\frac{1}{2} \le f'(x) \le 2$ dictates how quickly f may increase or decrease starting from $x = 0$ and ending at $x = 36$. So the smallest that $f(36)$ might be is $27 - \frac{1}{2}(36) = 9$ and the largest that $f(36)$ might be is $27 + 2(36) = 99$.

2. On the axes below, sketch the graph of a continuous function f that satisfies these conditions.

   $\lim\limits_{x\to-\infty} f(x) = 2 \quad f(3) = -1 \quad f'(3) = 0 \quad f''(3) < 0 \quad f(7) = 1 \quad f'(7) > 0 \quad f''(7) = 0 \quad \lim\limits_{x\to\infty} f(x) = \infty$



3. On the following plot featuring the graph of a function f, draw a small box $\square$ around each inflection point of the graph, and accurately sketch the graph of the derivative of f.



4. Consider the function f given by the formula $f(x) = \frac{x^2-6x+9}{1-2x}$. On what interval(s) is f increasing?

The function f will be increasing at any x for which $f'(x) > 0$. Taking the derivative of f,

$$f(x) = \frac{x^2-6x+9}{1-2x}$$

$$\Longrightarrow \quad f'(x) = \frac{(2x-6)(1-2x)-\left(x^2-6x+9\right)(-2)}{(1-2x)^2} = -2\frac{(x+2)(x-3)}{(1-2x)^2}.$$

Since $f'(x) = 0$ at $x = -2$ and $x = 3$ and $f'(x)$ is undefined at $x = \frac{1}{2}$, these three values correspond to critical values of f. Checking the sign of f' on the intervals between these critical values by testing a single input x, we see that

$$f'(-5) < 0 \qquad f'(0) > 0 \qquad f'(1) > 0 \qquad f'(5) < 0$$

and we can conclude that f is increasing on the intervals $\left(-2, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 3\right)$.

5. Consider the function f defined by the formula $f(x) = x^5 - x^2 + 1$.

   (a) What is the minimum output value of f on the closed interval $[0,1]$?

   The extrema of a continuous function on a closed intervals must occur at a critical value or an endpoint. Taking the derivative of the function, $f'(x) = 5x^4 - 2x$. Note that $f'(x) = 0$ at $x = 0$ and at $x = \sqrt[3]{2/5}$, and that

   $$f(0) = 1 \qquad f\left(\sqrt[3]{2/5}\right) \approx 0.6742698860086 \qquad f(1) = 1,$$

   indicating the minimum value of f on $[0,1]$ is approximately 0.6742698860086.

   (b) What is the maximum output value of f on the closed interval $[-1,2]$?

   Based on the same reasoning as the previous solution, any maximum must occur at $x = -1$ or $x = 0$ or $x = \sqrt[3]{2/5}$ or $x = 2$. Evaluating f at these values,

   $$f(-1) = -1 \qquad f(0) = 1 \qquad f\left(\sqrt[3]{2/5}\right) \approx 0.6742698860086 \qquad f(2) = 29,$$

   so we see that the maximum value of f on $[-1,2]$ is 29.

   (c) Does f have global extrema?

   No. Since the function is a fifth-degree polynomial with positive leading coefficient, $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, so f is unbounded above and below.

   (d) The graph of f has a single inflection point. What is the y-coordinate of this point?

   A graph has an inflection point where the graph changes concavity. The second derivative of our function is $f''(x) = 20x^3 - 2$, and $f''(x) = 0$ at $x = \sqrt[3]{1/10}$. Notably, on either side of this x-value, the second derivative has opposite sign, $f''(0) = -2$ and $f''(1) = 18$, which means the x-value $\sqrt[3]{1/10}$ does correspond to an inflection point. The y-coordinate of this inflection point will be $f\left(\sqrt[3]{1/10}\right) \approx 0.806100877897$.

   (e) What is an equation for the line tangent to the graph of f at its inflection point?

   The inflection point has approximate coordinates $(0.4641588833613, 0.806100877897)$. The slope of the line tangent to the graph at this point has slope $f'(0.4641588833613) \approx -0.6962383250419$, and so an equation for the tangent line is

   $$y - 0.806100877897 = -0.6962383250419(x - 0.4641588833613).$$

(f) Explain how the Intermediate Value Theorem *guarantees* f has a real root between -1 and 0.

Since f is a continuous function, and since $f(-1) = -1$ and $f(0) = 1$ (in particular f changes sign between -1 and 0) the Intermediate Value Theorem says f must have a root between -1 and 0.

(g) The function f has exactly one real root between -1 and 0. Its value is pretty close to $-\frac{4}{5}$, but not quite equal to $-\frac{4}{5}$. Demonstrate Newton's method by calculating a better approximation of this root. Try to calculate the value accurate to as many places to right of the decimal point as your calculator will afford you.

For $x_0 = -\frac{4}{5}$, the line tangent to the graph of f at x_0 has equation $y - f(x_0) = f'(x_0)\big(x - x_0\big)$. The next approximation x_1 for the root that Newton's method gives us is the x-intercept of this line, namely

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \approx -0.80885964912280701754\,.$$

For $x_1 = -0.80885964912280701754$, the line tangent to the graph of f at x_1 has equation $y - f(x_1) = f'(x_1)\big(x - x_1\big)$. The next approximation x_2 for the root that Newton's method gives us is the x-intercept of this line, namely

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx -0.80873062835888432464\,.$$

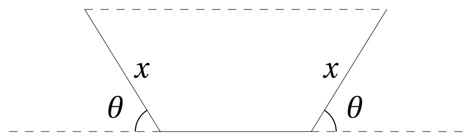
For $x_2 = -0.80873062835888432464$, the line tangent to the graph of f at x_2 has equation $y - f(x_2) = f'(x_2)\big(x - x_2\big)$. The next approximation x_3 for the root that Newton's method gives us is the x-intercept of this line, namely

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx -0.80873060047939331517\,.$$

Continuing in this manner, we may find ever-nearer approximations to the true root:

$-0.808730600479392013738554526511400064951377351559313075548116401836543340748321082\ldots$

6. A water trough with isosceles trapezoid-shaped cross-sections, similar to a typical rain gutter, is to be created from a long, one-foot wide strip of tin by taking a span of length x inches on either side of that width and bending each span up along the length of the strip by an angle θ, making the "sides" of the trough (see figure). If the sides are to be bent up by an angle of $\theta = \frac{\pi}{4}$, what choice of x maximizes the cross-sectional area, and thus the amount of water, the trough can hold?



The area of a trapezoid is the product of the average of the lengths of the parallel sides with the distance between those sides, i.e. the perpendicular height of the trapezoid. The length of the short side of the trapezoid is $1-2x$, with x measured in feet. The length of the long side of the trapezoid, the opening of the trough, is $1-2x+2x\cos(\theta) = 1-(2-\sqrt{2})x$. The perpendicular height of the trapezoid is $x\sin(\theta) = \frac{\sqrt{2}}{2}x$. Altogether then, the area of a cross-section of the trough, the function we seek to maximize, is defined by the formula

$$f(x) = \frac{(1-2x)+\left(1-(2-\sqrt{2})x\right)}{2}\left(\frac{\sqrt{2}}{2}x\right) = \frac{\sqrt{2}}{2}x - \left(\sqrt{2}-\frac{1}{2}\right)x^2 .$$

Extreme values may occur only at endpoints of the domain, where $x = 0$ (no trough sides) or $x = \frac{1}{2}$ (no trough bottom), or at a value of x where $f'(x) = 0$. Differentiating the function,

$$f(x) = \frac{\sqrt{2}}{2}x - \left(\sqrt{2}-\frac{1}{2}\right)x^2 \quad \Longrightarrow \quad f'(x) = \frac{\sqrt{2}}{2} - \left(2\sqrt{2}-1\right)x .$$

Letting $f'(x) = 0$ and solving for x, we get a critical value of $x = \frac{1}{4-\sqrt{2}}$. Evaluating f at the two endpoints and the one critical value,

$$f(0) = 0 \qquad f\left(\frac{1}{4-\sqrt{2}}\right) \approx 0.13672954017 \qquad f\left(\frac{1}{2}\right) = \frac{1}{8} = 0.125,$$

which indicates the maximal cross-sectional area is achieved for $x = \frac{1}{4-\sqrt{2}}$ feet, which is about 4.64 inches.

CHALLENGE: Suppose instead that x is fixed at five inches, so five inch spans will be folded up to make the "sides" of the trough, but that the angle θ may vary. In this case, what angle θ maximizes the amount of water the trough holds?

SUPER CHALLENGE: Suppose neither x nor θ are fixed; both may vary. Which pair(s) of values x and θ maximizes the amount of water the trough holds?

7. Suppose f is a cubic polynomial function with three real roots. The graph of f will have a single inflection point. Prove that the x-coordinate of the inflection point is the average of the three roots.

   Let r_1 and r_2 and r_3 denote the roots of f, and so the formula for f may be expressed as

   $$f(x) \;=\; a(x-r_1)(x-r_2)(x-r_3) \;=\; ax^3 - a(r_1+r_2+r_3)x^2 + a(r_1r_2+r_1r_3+r_2r_3)x - ar_1r_2r_3$$

   for some leading coefficient a. Calculating the first- and second-derivatives of f,

   $$f'(x) = 3ax^2 - 2a(r_1+r_2+r_3)x + a(r_1r_2+r_1r_3+r_2r_3) \qquad f''(x) = 6ax - 2a(r_1+r_2+r_3).$$

   The inflection point is located at the value of x for which $f''(x) = 0$;

   $$0 = 6ax - 2a(r_1+r_2+r_3) \quad\Longrightarrow\quad x = \frac{2a(r_1+r_2+r_3)}{6a} = \frac{r_1+r_2+r_3}{3},$$

   which is the average of the roots.

8. Three guests check into a hotel room. The manager says the bill is \$30, so each guest pays \$10. Later the manager realizes the bill should only have been \$25. To rectify this, he gives the bellhop \$5 as five one-dollar bills to return to the guests.

   On the way to the guests' room to refund the money, the bellhop realizes that he cannot equally divide the five one-dollar bills among the three guests. Since the guests are not aware of the total of the revised bill, the bellhop decides to just give each guest \$1 back and keep \$2 as a tip for himself, and proceeds to do so.

   Since each guest got \$1 back, each guest only paid \$9, bringing the total paid to \$27. The bellhop kept \$2, which when added to the \$27 comes to \$29. But if the guests originally handed over \$30, what happened to the remaining \$1?