Math135 Engineering Calculus I

The Last Pop Quiz

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NAME: ______________________________

1. Suppose that f is a *bounded* function: for all real numbers x, $\sqrt{3} \le f(x) \le 3$. Cross out all of the following numbers that *cannot* possibly be the value of $\int_2^5 f(x)\,\mathrm{d}x$.

$$\cancel{0} \quad \cancel{1} \quad \cancel{2} \quad \cancel{3} \quad \cancel{4} \quad \cancel{5} \quad 6 \quad 7 \quad 8 \quad 9 \quad \cancel{10} \quad \cancel{11} \quad \cancel{12} \quad \cancel{13} \quad \cancel{14} \quad \cancel{15} \quad \cancel{16} \quad \cancel{17} \quad \cancel{18} \quad \cancel{19} \quad \cancel{20}$$

If $f(x) = \sqrt{3}$ on the interval $(2,5)$, then $\int_2^5 f(x)\,\mathrm{d}x$ describes the area of a rectangle of area $3\sqrt{3} \approx 5.196$; this is the smallest that the value of the integral could be. If $f(x) = 3$ on the interval $(2,5)$, then $\int_2^5 f(x)\,\mathrm{d}x$ describes the area of a rectangle of area 9; this is the largest that the value of the integral could be.

2. Suppose that the function F is an antiderivative of f and that $F(1) = 3$. Of the following twelve expressions, *one* of them is equal to $F(9)$. Which one is it?

$$3 - f'(9) \qquad 1 + f'(9) \qquad f'(9) - f'(1) \qquad 3 + \int_1^9 f(x)\,\mathrm{d}x \qquad 1 + \int_3^9 f(x)\,\mathrm{d}x \qquad 9 + \int_1^3 f(x)\,\mathrm{d}x$$

$$f'(1) \qquad f'(9) \qquad f(9) - f(1) \qquad \int_3^9 f(x)\,\mathrm{d}x \qquad \int_1^9 f(x)\,\mathrm{d}x \qquad \int_1^3 f(x)\,\mathrm{d}x$$

$$\int_1^9 f(x)\,\mathrm{d}x = F(9) - F(1) \quad \Longrightarrow \quad \int_1^9 f(x)\,\mathrm{d}x = F(9) - 3 \quad \Longrightarrow \quad F(9) = 3 + \int_1^9 f(x)\,\mathrm{d}x$$

3. What is the result of evaluating these indefinite integrals?

$$\int \tfrac{t^7}{7} + \tfrac{7}{t^7} + \sqrt[7]{t} + 77\,\mathrm{d}t$$

$$\int \tfrac{t^7}{7} + \tfrac{7}{t^7} + \sqrt[7]{t} + 77\,\mathrm{d}t \;=\; \int \tfrac{1}{7}t^7 + 7t^{-7} + t^{1/7} + 77\,\mathrm{d}t \;=\; \tfrac{1}{56}t^8 - \tfrac{7}{6}t^{-6} + \tfrac{7}{8}t^{8/7} + 77t + C$$

$$\int \frac{x^2+7}{\sqrt{x-7}}\,\mathrm{d}x$$

Letting $u = x-7$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$, we have

$$\int \frac{(u+7)^2+7}{\sqrt{u}}\left(\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\right)\cancel{\mathrm{d}x} \;=\; \int \frac{u^2+14u+56}{\sqrt{u}}\,\mathrm{d}u \;=\; \int u^{3/2} + 14u^{1/2} + 56u^{-1/2}\,\mathrm{d}u$$

$$= \tfrac{2}{5}u^{5/2} + \tfrac{28}{3}u^{3/2} + 112u^{1/2} + C$$

$$= \tfrac{2}{5}(x-7)^{5/2} + \tfrac{28}{3}(x-7)^{3/2} + 112(x-7)^{1/2} + C.$$

(a) $\int \theta \sec^2(7\theta^2)\,\mathrm{d}\theta$

Letting $u = 7\theta^2$, so $\frac{1}{14}\frac{\mathrm{d}u}{\mathrm{d}\theta} = \theta$, we have

$$\int \left(\frac{1}{14}\frac{\mathrm{d}u}{\cancel{\mathrm{d}\theta}}\right)\sec^2(u)\,\cancel{\mathrm{d}\theta} \;=\; \frac{1}{14}\int \sec^2(u)\,\mathrm{d}u \;=\; \frac{1}{14}\tan(u) + C \;=\; \frac{1}{14}\tan(7\theta^2) + C.$$

4. What are the exact values of these definite integrals?

(a) $\displaystyle\int_{-7}^{7} x^3 \cos(x)\, \mathrm{d}x$

Since x^3 is an odd function and $\cos(x)$ is an even function, $x^3\cos(x)$ will be an odd function, and so since the domain of integration is symmetric about $x = 0$,

$$\int_{-7}^{7} x^3 \cos(x)\, \mathrm{d}x = 0\,.$$

(b) $\displaystyle\int_{0}^{8} \tfrac{1}{2} x \cos\left(x^2\right) \mathrm{d}x$

Letting $u = x^2$, so $\frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}x} = x$, we have

$$\int_{0}^{8} \tfrac{1}{2} x \cos\left(x^2\right) \mathrm{d}x \;=\; \int_{0}^{64} \tfrac{1}{2}\left(\tfrac{1}{2}\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\right) \cos(u)\, \cancel{\mathrm{d}x} \;=\; \tfrac{1}{4}\int_{0}^{64} \cos(u)\, \mathrm{d}u$$

$$= \tfrac{1}{4}\sin(u)\Big|_{0}^{64} \;=\; \tfrac{1}{4}\sin(64) \;\approx\; 0.230006509549197670837884$$

5. If $f'(x) = x^{\sqrt{3}}$ and if $f(12) = 325$, what must a formula for $f(x)$ be?

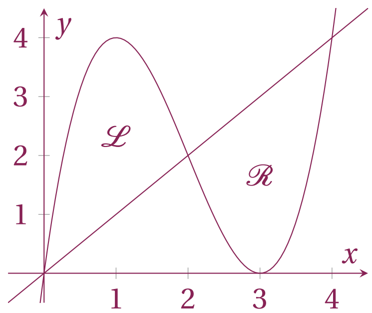
Since $f'(x) = x^{\sqrt{3}}$, the formula for the function f must have the form $f(x) = \frac{1}{\sqrt{3}+1} x^{\sqrt{3}+1} + C$ for some constant C. Since $f(12) = 325$,

$$325 = \tfrac{1}{\sqrt{3}+1}(12)^{\sqrt{3}+1} + C \quad\Longrightarrow\quad C \;=\; 325\frac{\sqrt{3}+1}{12^{\sqrt{3}+1}} \approx 1\,,$$

so $f(x) \approx \frac{1}{\sqrt{3}+1} x^{\sqrt{3}+1} + 1$

6. Consider the two functions f and g defined by the formulas $f(x) = x^3 - 6x^2 + 9x$ and $g(x) = x$. The graphs of these function bound two regions in the xy-plane; let $\mathscr{L}$ be the left-most region and let $\mathscr{R}$ be the right-most region.

(a) Sketch the graphs of f and g and label the regions $\mathscr{L}$ and $\mathscr{R}$.



The intersection points of the graphs can be found by solving the equation $f(x) = g(x)$:

$$\begin{aligned} & x^3 - 6x^2 + 9x = x \\ \implies\ & x^3 - 6x^2 + 8x = 0 \\ \implies\ & x(x-2)(x-4) = 0 \\ \implies\ & x = 0 \quad x = 2 \quad x = 4 \end{aligned}$$

(b) Show that the areas of regions $\mathscr{L}$ and $\mathscr{R}$ are equal.

$$\int_0^2 \left(x^3 - 6x^2 + 9x\right) - (x)\,\mathrm{d}x \;=\; \tfrac{1}{4}x^4 - 2x^3 + 4x^2\Big|_0^2 \;=\; \tfrac{1}{4}(2)^4 - 2(2)^3 + 4(2)^2 \;=\; 4$$

$$\int_2^4 (x) - \left(x^3 - 6x^2 + 8x\right)\mathrm{d}x \;=\; -\tfrac{1}{4}x^4 + 2x^3 - 4x^2\Big|_2^4$$

$$= \left(-\tfrac{1}{4}(4)^4 + 2(4)^3 - 4(4)^2\right) - \left(-\tfrac{1}{4}(2)^4 + 2(2)^3 - 4(2)^2\right) \;=\; 4$$

(c) Is the area of $\mathscr{L}$ and $\mathscr{R}$ being equal sufficient to conclude that the regions are congruent?

No. Two regions (shapes) must also have congruent boundaries to be congruent. While the linear boundary of $\mathscr{L}$ and $\mathscr{R}$ defined by g are clearly congruent since they have the same length, the cubic arcs of their boundaries defined by f are less obviously congruent. They *are* congruent, however. A sufficient argument would be show the graph of f is fixed under a half-rotation about the point $(2, 2)$, the cubic's inflection point.

(d) What is the volume of the solid generated by revolving $\mathscr{R}$ about the y-axis?

$$\int_2^4 (2\pi x)\Big((x)-\big(x^3-6x^2+9x\big)\Big)\,\mathrm{d}x \;=\; 2\pi\int_2^4 -x^4+6x^3-8x^2\,\mathrm{d}x$$

$$= 2\pi\Big(-\tfrac{1}{5}x^5+\tfrac{3}{2}x^4-\tfrac{8}{3}x^3\Big)\Big|_2^4 \;=\; \frac{368\pi}{15}$$

(e) What is the volume of the solid with base $\mathscr{L}$ whose cross-sections parallel to the y-axis are equilateral triangles?

A triangle with side-lengths A and B framing an angle θ has area $\frac{1}{2}AB\sin(\theta)$, so an equilateral triangle of side-length ℓ has area $\frac{\sqrt{3}}{4}\ell^2$. So the volume of our solid is

$$\int_0^2 \tfrac{\sqrt{3}}{4}\Big(\big(x^3-6x^2+9x\big)-(x)\Big)^2\,\mathrm{d}x \;=\; \tfrac{\sqrt{3}}{4}\int_0^2 x^6-12x^5+52x^4-96x^3+64x^2\,\mathrm{d}x$$

$$= \tfrac{\sqrt{3}}{4}\Big(\tfrac{1}{7}x^7-2x^6+\tfrac{52}{5}x^5-24x^4+\tfrac{64}{3}x^3\Big)\Big|_0^2$$

$$= 32\sqrt{3}\Big(-\tfrac{6}{7}+\tfrac{13}{5}-3+\tfrac{4}{3}\Big) \;=\; \frac{256\sqrt{3}}{105}$$

7. A 0.6 kg model rocket is loaded up with 1.2 kg of rocket fuel which, upon launch, straight upward, it burns through at a rate of 0.004 kg/m (kg per meter of altitude). How much work does the burning rocket fuel do propelling the rocket upward between the moment it launches and the moment the fuel is completely expended? Use $9.797\,\mathrm{m/s^2}$ for the force of gravity around Grand Junction.

   The rocket expends all its fuel after having climbed $1.2/0.004 = 300$ meters. The amount of work done is the integral of the force applied lifting the rocket against the force of gravity with respect to the displacement (altitude) x. Note that the mass of the *remaining* rocket fuel after the rocket has climbed x meters is $1.2 - 0.004x$.

$$\text{Work} = \int \text{Force}\,\mathrm{d}x = \int ma\,\mathrm{d}x = \int_0^{300} \big(0.6 + (1.2 - 0.004x)\big)(9.797)\,\mathrm{d}x = \cdots$$

$$\cdots = 9.797\int_0^{300} 1.8 - 0.004x\,\mathrm{d}x = 9.797\Big(1.8x - 0.002x^2\Big)\Big|_0^{300} = 3526.92\,\text{Joules}$$

8. There are five houses of different colors next to each other on the same road. In each house lives a woman of a different nationality. Each woman has her favorite drink, her favorite brand of cigarettes, and keeps a particular kind of pet.

   The Englishwoman lives in the red house.
   The Swede keeps dogs.
   The Dane drinks tea.
   The green house is just to the left of the white one.
   The owner of the green house drinks coffee.
   The Pall Mall smoker keeps birds.
   The owner of the yellow house smokes Dunhills.
   The woman in the center house drinks milk.
   The Norwegian lives in the first house.
   The Blend smoker has a neighbor who keeps cats.
   The woman who smokes Blue Masters drinks bier.
   The woman who keeps horses lives next to the Dunhill smoker.
   The German smokes Prince.
   The Norwegian lives next to the blue house.
   The Blend smoker has a neighbor who drinks water.

   Who keeps fish?