The Free Particle

$\hat{H}\psi = E\psi \qquad V=0 \qquad -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E\psi$

$\psi'' = -\frac{2mE}{\hbar^2}\psi \quad \rightarrow \quad \psi'' + \frac{2mE}{\hbar^2}\psi = 0 \qquad (D+ik)(D-ik)\psi = 0$

$k = \sqrt{\frac{2mE}{\hbar^2}} \quad \rightarrow \quad \psi = \underset{\rightarrow}{Ae^{ikx}} + \underset{\leftarrow}{Be^{-ikx}}$

E is any positive number

$\Psi(x,t) = Ae^{i(kx - \frac{\hbar k^2}{2m}t)} + Be^{-i(kx + \frac{\hbar k^2}{2m}t)}$

$E = \frac{\hbar^2 k^2}{2m} \qquad \frac{E}{k} = \frac{\hbar k^2}{2m}$

Now $F(x \pm vt) \rightarrow$ wave in the $\mp x$ direction, speed $= v$

Now $v = \lambda\nu \qquad 2\pi\nu = \omega \qquad \omega = \frac{\hbar k^2}{2m} \qquad \nu = \hbar\frac{(2\pi)^2}{\lambda^2}\cdot\frac{1}{2m}\cdot\frac{1}{2\pi}$

$\nu = \frac{\hbar 2\pi}{2m\lambda^2} \qquad v = \frac{2\pi\hbar}{2m\lambda}$

$x \pm vt =$ constant

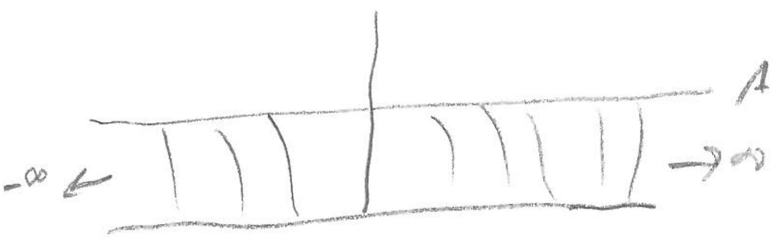
So $x = \mp vt +$ constant → Shape does not change

Now $v_{quantum} = \frac{\hbar k}{2m} = \sqrt{\frac{E}{2m}}$

$v_{classical} = \sqrt{\frac{2E}{m}} = 2 v_{quantum}$

? $v_q = \frac{1}{2} v_c$? Later

have Ψ Now normalize → $A^2 \int_{-\infty}^{+\infty} e^{ikx} e^{-ikx} dx = \int_{-\infty}^{\infty} 1 \, dx$ which



$= \infty$ NOPE

Previously, for quantized solutions $\Psi(x,t) = \sum c_n \psi_n(x) e^{-i\frac{E_n t}{\hbar}}$

$c_n = \int \Psi(x,0) \psi_m^* dx$ So $\Psi(x,t) = \sum \left[\int \Psi(x,0) \psi_m^* dx \right] \psi_n(x) e^{-i\frac{E_n t}{\hbar}}$

Now Ψ function of continuous E not $\underline{E_n}$

So $\sum \to \int$ and $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m} t)} dk$

where $k \in [-\infty, +\infty]$

Now $\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{ikx}\, dx$

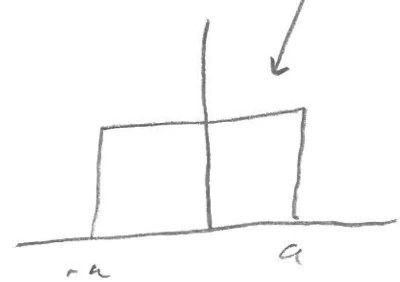
$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Psi(x,0)\, e^{-ikx}\, dx$ → fourier Transform

---

$$\Psi(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \Psi(x',0)\, e^{ikx'}\, \underline{\underline{dx'}} \right] e^{i(kx' - \frac{\hbar k^2}{2m} t)}\, \underline{\underline{dk}}$$

Square pulse

Now $\Psi(x,0)$ exists and is normalizable



Example → $\Psi(x,0) = \begin{cases} A & x \in [-a, a] \\ 0 & \text{else} \end{cases}$

$\Psi(x,t) = ?$ #1, A ? $\int_{-a}^{a} A^2\, dx = A^2 x \Big|_{-a}^{a} = A^2(a-(-a)) = 2aA^2 = 1$

$A = \frac{1}{\sqrt{2a}}$ → $\Psi(x,0) = \frac{1}{\sqrt{2a}}$ $x \in [-a, a]$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{+a} \underbrace{\frac{1}{\sqrt{2a}}}_{\Psi(x,0)} e^{-ikx}\, dx = \frac{1}{2\sqrt{\pi a}} \int_{-a}^{+a} e^{-ikx}\, dx$$

$$= \frac{1}{2(\pi a)^{1/2}} \frac{e^{-ikx}}{(-ik)}\Big|_{-a}^{a} = \frac{1}{2i(\pi a)^{1/2}k} e^{-ikx}\Big|_{a}^{-a} = \frac{1}{2i(\pi a)^{1/2}k}\left[e^{+ika} - e^{-ika}\right]$$

$$= \frac{1}{(\pi a)^{1/2}}\left[\frac{e^{ika} - e^{-ika}}{2ik}\right] = \frac{1}{\sqrt{\pi a}} \cdot \frac{\text{SIN}(ka)}{k}$$

$$= \frac{1}{\sqrt{\pi a}} \cdot a\left[\frac{\text{SIN}(ka)}{ka}\right] = \sqrt{\frac{a}{\pi}}\ \text{SINC}(ka)$$

So $\phi(k) = \sqrt{\frac{a}{\pi}}\ \text{SINC}(ka)$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty \leftarrow \text{not } a} \underbrace{\sqrt{\frac{a}{\pi}}\ \text{SINC}(ka)}_{\phi(k)}\ e^{i(kx - \frac{\hbar k^2}{2m}t)}\,dk$$

you build x from k

CAN NOT BE EVALUATED Analytically

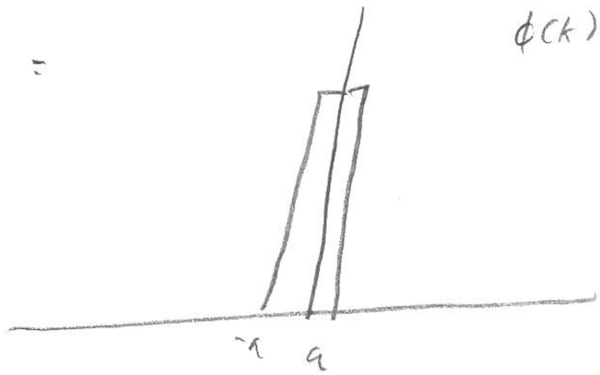
For $ka \ll 1$ a small $\rightarrow$ $\text{SIN}(ka) \sim ka$ $\quad \text{SIN}(x) \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

So $\phi(k) \sim \sqrt{\frac{a}{\pi}} \cdot \frac{ka}{ka} \sim \sqrt{\frac{a}{\pi}}$ or $\underline{\phi(k) \text{ is Flat}}$

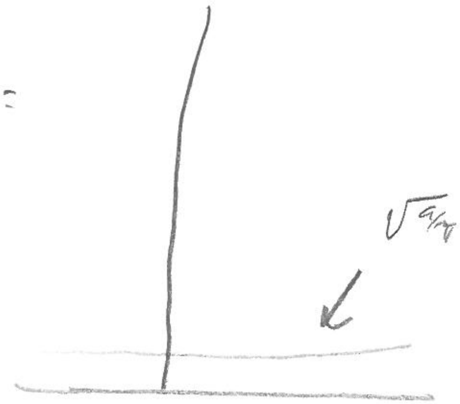
Corresponds to small Δx

Solve with integrator numerically

$\Psi(x,0)$ =



$\phi(k)$ =



Localized x = infinite k's → p's

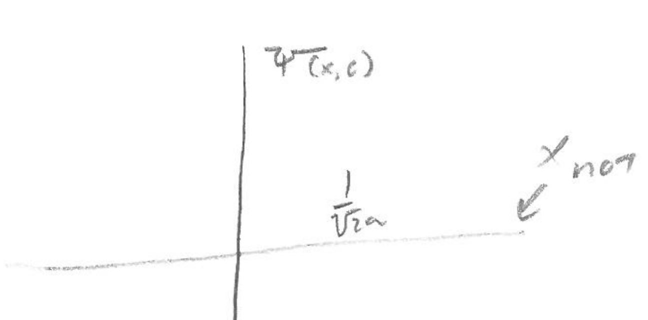
Another way of thinking

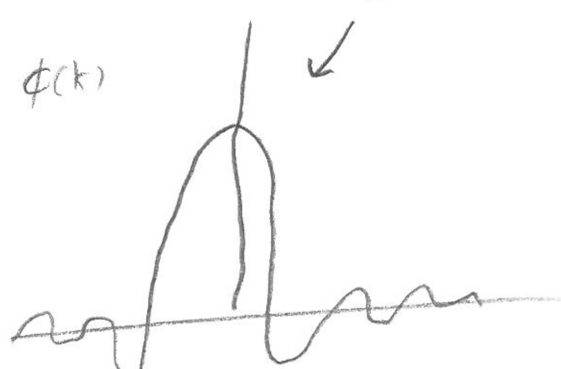
$\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \rightarrow$

$\hat{p}$ → well constrained

x → Not

Conversely $x \in [-a, a]$ a <u>large</u>

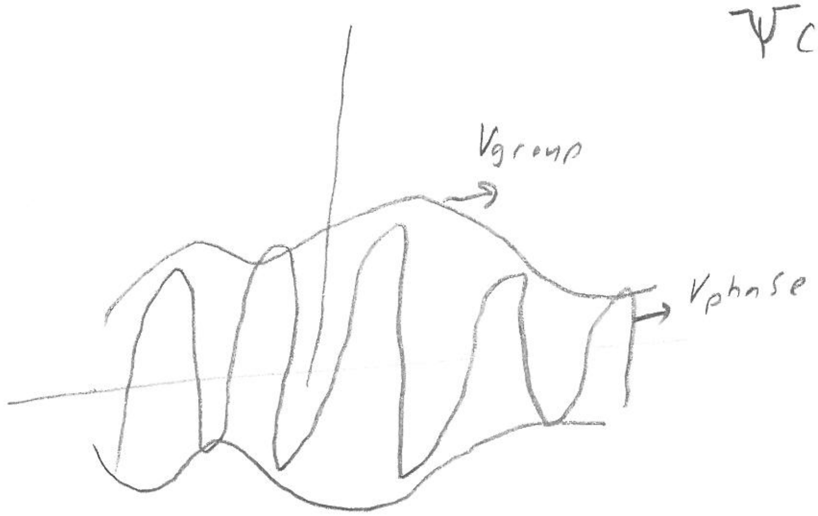




So → we build wave packets from waves with continuous k's moving at various velocities $\lambda \nu = v$ $\frac{\nu}{2\pi} = \omega$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{i(kx-\omega t)}\, dk$$

how much from each k



$$\omega = \frac{\hbar k^2}{2m} \qquad k \uparrow\downarrow\ \omega \uparrow\downarrow$$

components move differently

whole moves at v_{group}

$\omega = \omega(k)$ → dispersion relationship

Assume $\phi(k)$ peaked at k_0 $\omega(k) \sim \underset{\frac{\hbar k_0^2}{2m}}{\omega_0} + \underset{\omega_0'}{\left.\frac{d\omega}{dk}\right|_{k_0}} (k-k_0)$ Taylor Expansion

$s = k - k_0 \quad ds = dk$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(s+k_0)\, e^{i[\overset{k}{(s+k_0)}x - \overset{\omega}{(\omega_0 + \omega_0' s)}t]}\, ds$$

$$\sim \frac{1}{\sqrt{2\pi}} e^{i(-\omega_0 t + k_0 \omega_0' t)} \int_{-\infty}^{+\infty} \phi(s+k_0)\, e^{i(k_0+s)(x-\omega_0' t)}\, ds$$

$$\sim e^{-i(\omega_0 - k_0 \omega' t)}\, \Psi(x - \omega_0' t, 0)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s)\, e^{i(k_0+s)x}\, ds$$

$$\text{Speed} = \frac{d\omega}{dk} = \frac{2\hbar k}{2m} = \frac{\hbar k}{m} = \frac{p}{m} = V_{classical}$$

$$V_g = 2\,V_{phase}$$