Formalism → Uncertainty and dirac notation

$$\sigma_a^2 \sigma_b^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

Ex $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$ $\quad \sigma_x^2 \sigma_p^2 = \left(\frac{1}{2i} \frac{i\hbar}{1} \right)^2 = \left(\frac{\hbar}{2} \right)^2$

Identity $[AB, C] = A[B, C] + [A, C]B$

$\rightarrow = ABC - CAB = (ABC - ACB + ACB - CAB)$

and $[A, B+C] = [A, B] + [A, C]$

For any Non-Commuting observables you can NEVER measure the 2 to a better accuracy than given by the above.

Example → you localize an electron to $\sigma_x \sim$ diameter H atom $\sim 10^{-10}$ meters. What is the best estimate on momentum and velocity?

$\sigma_x \sigma_p \geq \frac{\hbar}{2}$ $\quad \sigma_p \sim mv$ $\quad \sigma_p \geq \frac{\hbar}{2\sigma_x}$ $\quad m = 9.11 \cdot 10^{-31}\ kg$

$\hbar = 1.05 \cdot 10^{-34}\ m^2 \frac{kg}{s}$

$$\sigma_p \geq \frac{1.05 \cdot 10^{-34}}{2 \cdot 10^{-10}} \frac{kg\,m}{s} \sim .5 \cdot 10^{-24}\ kg \frac{m}{s} = 5 \cdot 10^{-25}\ kg \frac{m}{s}$$

$$\sigma_v \geq \frac{5 \cdot 10^{-25}}{9.11 \cdot 10^{-31}} \frac{m}{s} \sim \frac{1}{2} \cdot 10^{6} \frac{m}{s} \sim 5 \cdot 10^{5} \frac{m}{s}$$

minimum uncertainty wave packet → gaussian

Skip details

Evolution of Expectation Value in Time

$$\frac{d}{dt}\langle \hat{Q} \rangle = \frac{d}{dt}\langle \Psi | \hat{Q}\Psi \rangle = \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t}\Psi \right\rangle$$

$$+ \left\langle \Psi \middle| \hat{Q}\frac{d\Psi}{dt} \right\rangle$$

but $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi \rightarrow \frac{\partial \Psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\Psi$

So $$\frac{d\langle \hat{Q} \rangle}{dt} = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi \rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

$\hat{H}$ is hermitian so $= -\frac{1}{i\hbar}\langle \Psi | \hat{H}\hat{Q}\Psi \rangle$

So $$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{i}{\hbar}\langle \Psi | \hat{H}\hat{Q}\Psi \rangle - \frac{i}{\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

$$= \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \left\langle \frac{\partial \hat{Q}}{\partial t} \right\rangle$$

$\hat{Q} = x$?

$$\frac{d}{dt}\langle x\rangle = \langle \underbrace{\frac{\partial x}{\partial t}}_{0} \rangle + \frac{i}{\hbar}\langle [\hat{H}, \hat{x}] \rangle$$

$$[\hat{x}, \hat{H}] = [x, \frac{p^2}{2m} + V] = \frac{1}{2m}[x, \frac{p^2}{2m}] + [x, V]$$

$$= \frac{1}{2m}\left(xp^2 - pxp + pxp + p^2x\right) = \frac{1}{2m}[x,p]p + p[x,p]$$

$$= \frac{1}{2m}(i\hbar p + pi\hbar) = \frac{i\hbar p}{m} \quad \text{so} \quad \langle [\hat{H}, x] \rangle = -\frac{i\hbar\langle p\rangle}{m} \quad \text{as}$$

$$[\hat{x}, \hat{V}] = xV - Vx = 0$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\left(-\frac{i\hbar\langle p\rangle}{m}\right) = \frac{\langle p\rangle}{m} \quad \text{and}$$

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\frac{i\hbar}{m}\langle p\rangle\right)^2 = \left(\frac{\hbar}{2m}\langle p\rangle\right)^2$$

Now if $\hat{Q}$ not $\hat{Q}(t)$ and $[\hat{H}, \hat{Q}] = 0$ Then

$\langle \hat{Q} \rangle$ = Constant, conserved

Now $\dfrac{d\langle \hat{Q} \rangle}{dt} = \dfrac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \left\langle \dfrac{\partial \hat{Q}}{\partial t} \right\rangle$

if $\hat{Q}$ not $\hat{Q}(t)$ $\quad \dfrac{d\langle \hat{Q} \rangle}{dt} = \dfrac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle$

So $\sigma_H^2 \sigma_Q^2 \geq \left(\dfrac{1}{2i} \langle [\hat{H}, \hat{Q}] \rangle \right)^2 = \left(\dfrac{1}{2i} \dfrac{\hbar}{i} \dfrac{d\langle \hat{Q} \rangle}{dt} \right)^2$

$= \left(\dfrac{\hbar}{2} \right)^2 \left(\dfrac{d\langle \hat{Q} \rangle}{dt} \right)^2$ or $\sigma_H \sigma_Q \geq \dfrac{\hbar}{2} \left| \dfrac{d\langle \hat{Q} \rangle}{dt} \right|$

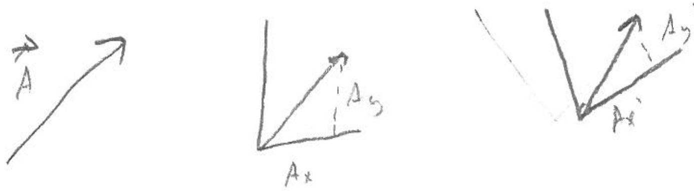
Let $\Delta E = \sigma_H$ and $\Delta t = \dfrac{\sigma_Q}{|d\langle \hat{Q} \rangle / dt|}$

Then $\sigma_Q = \left| \dfrac{d\langle \hat{Q} \rangle}{dt} \right| \Delta t$ ✓ Time for $\langle \hat{Q} \rangle$ to change

by one standard deviation

and $\Delta E \, \Delta t \geq \dfrac{\hbar}{2}$

Dirac Notation



$\vec{A}$ still $\vec{A}$

$\Psi(x,t) = \langle x | A(t) \rangle$ $\quad \Phi(p,t) = \langle p | A(t) \rangle$

For discrete $\rightarrow$ $c_n(t) = \langle n | A(t) \rangle$

all same $\vec{A}$, same info

$\Psi(x,t) = \int \Psi(y,t)\,\delta(x-y)\,dy = \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p,t)\, e^{ipx/\hbar}\, dp$

$= \sum c_n\, e^{-iE_n t/\hbar}\, \psi_n$

Operators just take $|\alpha\rangle \rightarrow \hat{Q}|\alpha\rangle$ to $|\beta\rangle$

with $|\alpha\rangle = \sum a_n |e_n\rangle$
just one basis

and $a_n = \langle e_n | \alpha \rangle$

$|\beta\rangle = \sum b_n |e_n\rangle \Rightarrow b_n = \langle e_n | \beta \rangle$

$Q_{mn} = \langle e_m | \hat{Q} | e_n \rangle$

So $\sum b_n |e_n\rangle = \sum a_n \hat{Q} |e_n\rangle$

Or $\sum_n b_n \underbrace{\langle e_m | e_n \rangle}_{\delta_{mn}} = \sum_n a_n \langle e_m | \hat{Q} | e_n \rangle$

$b_m = \sum_n Q_{mn} a_n$

Ex) $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|A\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ $|a|^2 + |b|^2 = 1$

Let $\hat{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$ at $T = 0$ $|A\rangle = |1\rangle$ $T \neq 0$?

$i\hbar \frac{d}{dt} |A\rangle = \hat{H} |A\rangle$ Start with $\hat{H}|A\rangle = E|A\rangle$

!

$$\left|\begin{pmatrix} h-E & g \\ g & h-E \end{pmatrix}\right| \to (h-E)^2 - g^2 = 0$$

$$h-E = \pm g \qquad E = h+g\ ,\ h-g$$

$$\begin{pmatrix} h & g \\ g & h \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = h \pm g \begin{pmatrix} a \\ b \end{pmatrix}$$

$$ha + gb = (h \pm g)a \qquad B = \pm a$$

$$|A_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

$$|A(0)\rangle = (|1\rangle) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|A_+\rangle + |A_-\rangle\right) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|A(t)\rangle = \frac{1}{2}\left[e^{-i(h+g)t/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-i(h-g)t/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$= \frac{1}{2} e^{-\frac{iht}{\hbar}}\left[e^{-igt/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{igt/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$= e^{-iht/\hbar}\begin{bmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{bmatrix}$$

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix} \qquad \langle\alpha| = \left(a_1^*, a_2^* \cdots \right)$$

$\hat{P}$ $\quad |\alpha\rangle\langle\alpha|$ $\qquad \langle\alpha|\beta\rangle|\alpha\rangle$ $\quad$ n dimensional Projection operator

$\#$ $\qquad$ vector

For $\quad \langle e_m|e_n\rangle = \delta_{mn} \qquad \sum_n |e_n\rangle\langle e_n| = 1$

$\quad \langle e_z|e_{z'}\rangle = \delta(z-z') \qquad \int |e_z\rangle\langle e_z| = 1$

$$\sum_n |e_n\rangle\langle e_n|\alpha\rangle = |\alpha\rangle$$

Ex $\quad |\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle \qquad |\beta\rangle = i|1\rangle + 2|3\rangle$

a.) $\quad -i\langle 1| - 2\langle 2| + i\langle 3| \qquad -i\langle 1| + 2\langle 3|$

$\langle\alpha|\beta\rangle = -ii\langle 1|1\rangle + i2\langle 3|3\rangle = 1+2i$

$\langle\beta|\alpha\rangle = -ii\langle 1|1\rangle + 2(-i)\langle 3|3\rangle$

$= 1-2i$

$\hat{A} = |\alpha\rangle\langle\beta| \qquad \hat{A}_{11} = \langle 1|\alpha\rangle\langle\beta|1\rangle = i\langle 1|1\rangle(-i)\langle 1|1\rangle = 1$

$\hat{A}_{23} = \langle 2|\alpha\rangle\langle\beta|2\rangle = -2\langle 2|2\rangle 2\langle 3|3\rangle = -4$

$\hat{A}_{33} \rightarrow \langle 3|\alpha\rangle\langle 3|\beta\rangle = -i2 = -2i$

$$\hat{A} = \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

Ex.)

$$\hat{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix} \quad \hat{H}\psi = E\psi \quad \begin{vmatrix} a-E & 0 & b \\ 0 & c-E & 0 \\ b & 0 & a-E \end{vmatrix} = 0$$

$$(a-E)(c-E)(a-E) - b \cdot b \cdot (c-E) = 0$$

$$((a-E)^2 - b^2)(c-E) = 0 \quad E = c, \quad a-E = \pm b \quad E = a+b \quad E = a-b$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad |3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$E = c \quad \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = c\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$aa_1 + ba_3 = ca_1 \quad (c-a)a_1 = ba_3 \rightarrow 0$

$ca_2 = ca_2 \rightarrow 1$

$ba_1 + aa_3 = ca_3 \quad (c-a)a_3 = -ba_1 \rightarrow 0$

$$a+b = E \quad \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a+b\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$aa_1 + ba_3 = aa_1 + ba_1$

$ba_3 = ba_1 \quad a_3 = a_1$

$ca_2 + (a+b)a_2$ No

$$E = a-b \quad \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a-b)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$aa_1 + ba_3 = aa_1 - ba_1$

$ba_3 = -ba_1$

$a_3 = -a_1$

$ca_2 = (a-b)a_2 \rightarrow 0$

$|1\rangle, |2\rangle, |3\rangle \rightarrow$ complete set

$$|S(0)\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} = c_1\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{c_3}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \sum_1^3 c_n|\alpha_n\rangle$$

$$+\frac{c_2}{\sqrt{2}} + \frac{c_3}{\sqrt{2}} = 0 \qquad c_2 = -c_3 \rightarrow \text{Pick } 0$$

$$c_1 = 1$$

$$|S(0)\rangle = |1\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |S(T)\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}e^{-ict/k}$$

$$|S(0)\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

$$c_1\begin{pmatrix}0\\1\\0\end{pmatrix} + \frac{c_2}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{c_3}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\frac{c_2}{\sqrt{2}} + \frac{c_3}{\sqrt{2}} = 1 \qquad c_1 = 0 \qquad \frac{c_2}{\sqrt{2}} - \frac{c_3}{\sqrt{2}} = 0$$

$$\therefore c_2 = c_3 \qquad \frac{2c_2}{\sqrt{2}} = 1 \qquad c_2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$|S(0)\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right)$$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}\left[|2\rangle e^{-i(a+b)t/\hbar} + |3\rangle e^{-i(a-b)t/\hbar} \right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-i(a+b)t/\hbar} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-i(a-b)t/\hbar} \right]$$

$$= \frac{e^{-iat/\hbar}}{2}\begin{pmatrix} e^{-ibt/\hbar} + e^{ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} - e^{+ibt/\hbar} \end{pmatrix}$$

$$= e^{-iat/\hbar}\left[\begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i\sin(bt/\hbar) \end{pmatrix}\right]$$

For $|S(0)\rangle = |S_1\rangle$ $P(c), P(a-b)\ P(a+b)$?

$\frac{1}{\sqrt{2}}\left(|S_2\rangle + |S_3\rangle\right)$ $P(c), P(a-b), P(a+b)$?

$$c_n = \underline{\underline{\langle e_n|\Psi\rangle}}$$

Let $A = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad \lambda_n = 2\lambda,\ \lambda,\ -\lambda$

$|1'\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad |2'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad |3'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$|S(0)\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ old basis

$P(2\lambda)\ ? = \langle 1' | S(0) \rangle = \overline{0\ 0\ 1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

$P(\lambda)\ ? = \langle 2' | S(0) \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \overline{1\ 1\ 0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \cdot (2) = 1$

$P(-\lambda) = \langle 3' | S(0) \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \overline{1\ {-1}\ 0} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1-1}{2} = 0$

$\sum |C_n|^2 = 1$ ✓