Formalism

Wave Functions and Operators

Wave functions satisfy the conditions for abstract vectors $|\alpha\rangle$

Operators " " matrices $(\ \ ) \rightarrow \hat{Q}$

Wave function solutions, when, properly normalized form an orthonormal basis $|\psi_i\rangle \rightarrow \langle\psi_j|\psi_i\rangle = \delta_{ji}$ where $|\psi_i\rangle \rightarrow |e_i\rangle$

In n dimensional space $|\psi\rangle \rightarrow |\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix} = \sum a_i |e_i\rangle$

$\langle\alpha|\beta\rangle = a_1^* b_1 + a_2^* b_2 + \cdots a_n^* b_n$

Inner Product

$|\psi\rangle \rightarrow$ must be normalized, in function space $\langle\psi|\psi\rangle = 1$

$\rightarrow \int \psi^* \psi \, dx = 1$ so $\psi(x) \rightarrow 0$ $x \rightarrow \pm\infty$ $\psi(x) \in$ Hilbert Space

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We define the inner product of $\langle f|g\rangle = \int f^* g \, dx$

with the property that $\langle g|f\rangle = \langle f|g\rangle^*$ and $\langle f|f\rangle = \int f^* f \, dx$

Properly normalized f's satisfy $\langle f_i | f_j \rangle = \delta_{ij}$

$\rightarrow$ orthonormal set

and any $f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \rightarrow \quad |\alpha\rangle = \sum a_i |e_i\rangle$

For orthonormal $f_n \rightarrow \quad c_n = \langle f_n | f \rangle = \int f_n^* f(x)\, dx$

Ex. $\quad f = f^{\nu} \quad x \in [0,1] \quad \langle f | f \rangle = \int_0^1 x^{2\nu}\, dx = \left. \dfrac{x^{2\nu+1}}{2\nu+1} \right|_0^1$

$= \dfrac{1}{2\nu+1} - 0$ good if $2\nu + 1 > 0$ or $\nu > -\frac{1}{2}$

Observables

$\hat{p} = -i\hbar \dfrac{d}{dx} \qquad \hat{S}^2 = \dfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Expectation Values $\quad \langle \psi | \hat{Q} | \psi \rangle = \int \psi^* \hat{Q} \psi\, dx$

$= \int \psi^* \hat{p} \psi\, dx \ , \quad \boxed{a^* b^* c^*} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \dfrac{3}{4}\hbar^2$

An observable is real $\langle Q \rangle = \langle Q \rangle^*$

or $\langle \psi | \hat{Q} | \psi \rangle = \langle \hat{Q} \psi | \psi \rangle$ moving $\hat{Q}$ to the Bra

Changes it: From $\int \psi^* \hat{Q} \psi \, dx \to \int (\hat{Q} \psi)^* \psi \, dx$

Hermetian!!

Observables are represented by <u>hermetian</u> <u>operators</u>

For matrices check if $\hat{Q} = \left((\hat{Q})^T \right)^*$

For observables like $\hat{p} = -\hbar i \frac{d}{dx}$?

$$\langle \psi | \hat{p} \psi \rangle = \int_{-\infty}^{+\infty} \psi^* \left[(-\hbar i) \frac{d}{dx} \psi \right] dx \qquad uv\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} v \, du = \int_{-\infty}^{+\infty} u \, dv$$

$$\int_{-\infty}^{+\infty} \psi^* (-\hbar i) \frac{d}{dx} \psi \, dx = (-\hbar i) \underbrace{\psi^* \psi \Big|_{-\infty}^{+\infty}}_{0} - \int (-\hbar i) \frac{d\psi^*}{dx} \psi \, dx$$

$u = \psi^*$ $dv = \frac{d\psi}{dx} dx$

$du = \frac{d\psi^*}{dx} dx$ $v = \psi$

$$= \int_{-\infty}^{+\infty} (\hbar i) \frac{d\psi^*}{dx} \psi \, dx = \int_{-\infty}^{+\infty} \left(-\hbar i \frac{d\psi}{dx} \right)^* \psi \, dx = \langle \hat{p} \psi | \psi \rangle \checkmark$$

Determinate States

measure $\hat{Q}|\Psi\rangle$ get $q|\Psi\rangle$ if $|\Psi\rangle = |\Psi_n\rangle$

Then $\hat{Q}|\Psi_n\rangle = q_n|\Psi_n\rangle$ will ALWAYS get Same

Value on identical Systems

No deviation $\sigma^2 = \langle(\hat{Q} - \langle Q\rangle)^2\rangle$

since $\langle Q\rangle = q$

$= \langle\Psi|(\hat{Q} - q)^2\Psi\rangle = \langle(\hat{Q} - q)\Psi|(\hat{Q} - q)\Psi\rangle$

$= \langle(q-q)\Psi|(q-q)\Psi\rangle = \underline{\underline{0}}$

SE requires $\hat{H}\Psi = E\Psi$ So E is a determinate State

Collection of ALL eigenvalues is the Spectrum

if one q_n belongs to MORE than one Ψ_n the Spectrum

is degenerate

Consider $\hat{Q} = i\frac{d}{d\phi}$ is this hermetian?

$$\langle \psi | \hat{Q} \psi \rangle = \int \psi^* i\frac{d}{d\phi} \psi \, d\phi \quad \text{where} \quad \psi(\phi + 2\pi) = \psi(\phi)$$

ψ 1. is it hermetian? 2. what are ψ_n's

3. what are q_ns ?

$$i\int \psi^* \frac{d\psi}{d\phi} d\phi = \underbrace{i(\psi\psi^*)\Big|_0^{2\pi}}_{0} - i\int \frac{d\psi^*}{dx}\psi \, dx = +\int \left(i\frac{d\psi}{d\phi}\right)^* \psi \, dx$$

$$= \langle \hat{Q}\psi | \psi \rangle \checkmark$$

2. $i\frac{d\psi}{d\phi} = q\psi \rightarrow -i\ln(\psi) = q\phi + C$

$$\ln(\psi) = -iq\phi + D$$

$\psi(\phi) = Ae^{-iq\phi}$ but $Ae^{-iq(\phi+2\pi)} = Ae^{-iq\phi}$

So $Ae^{-iq2\pi} = A$ or $e^{-iq2\pi} = 1$ So $q = \pm n, 0$

Spectrum $= 0, \pm 1, \pm 2 \ldots\ldots$ non-degenerate