Linear Algebra Continued

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{matrix} 3\cdot 1 + 4\cdot 2 + 5\cdot 3 \\ 1\cdot 1 + 2\cdot 2 + 3\cdot 3 \\ 0\cdot 1 + 1\cdot 2 + 3\cdot 4 \end{matrix} = \begin{matrix} T_{11}\alpha_{11} + T_{12}\alpha_{21} + T_{13}\alpha_{31} \\ T_{21}\alpha_{11} + T_{22}\alpha_{21} + T_{23}\alpha_{31} \\ T_{31}\alpha_{11} + T_{32}\alpha_{21} + T_{33}\alpha_{31} \end{matrix}$$

Transform

Aside from Multiplication need to know Transpose, conjugate, Determinants

inverses, how to get eigenvalues and eigenvectors

I. Determinate $|(A)| = \#$

2x2 $\left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right| = AD - BC$ $\left|\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right| = 4 - 6 = -2$

3x3 $\left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right|$ Put finger on a_{11} cross out row/column

multiply by $(-1)^{i+j}$, repeat

$$a(ei-hf)(-1)^2 + b(di-fg)(-1)^3 + c(dh-eg)(-1)^4$$

Ex.

$$\begin{vmatrix} 1 & -5 & 2 \\ 7 & 3 & 4 \\ 2 & 1 & 5 \end{vmatrix}$$

$$1\cdot(3\cdot5-1\cdot4)-(-5)(7\cdot5-4\cdot2)+2(7\cdot1-3\cdot2)$$

$$(15-4)+5(35-8)+2(7-6)$$

$$11+(175-40)+2$$

$$=148$$

Conjugate $A \to i \to -i \to A^*$

Transpose $\to$ Switch rows and columns A^T ← transpose

$$[1,2]^T=\begin{bmatrix}1\\2\end{bmatrix} \quad \begin{bmatrix}1&2\\3&4\end{bmatrix}^T=\begin{pmatrix}1&3\\2&4\end{pmatrix} \quad \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}^T=\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$$

Inverses

$$AA^{-1}=\mathbb{1}=\begin{pmatrix}1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\cdots\\ \vdots & & & \ddots\end{pmatrix}$$

only diagonal $\neq 0$

2D $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ $\underline{A^{-1}=\frac{1}{\det(A)}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}}$

$$A=\begin{pmatrix}2&3\\4&5\end{pmatrix} \quad A^{-1}=\frac{1}{10-12}\begin{pmatrix}5&-3\\-4&2\end{pmatrix}$$

$$AA^{-1}=\frac{1}{-2}\begin{pmatrix}2&3\\4&5\end{pmatrix}\begin{pmatrix}5&-3\\-4&2\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}-2&0\\0&-2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$A^{-1}(3\times 3) = \frac{1}{|A|}(\text{adjoint} A) = \frac{1}{|A|}(C_{ij})^T$$

$$C_{ij} = (-1)^{i+j} M_{ij}$$ ← Cofactor Matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{pmatrix} \qquad |A| = 1\cdot(24) - 2(-5) + 3(-4) = 22 \checkmark$$

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} (4\cdot 6 - 0\cdot 5) & -(0\cdot 6 - 5\cdot 1) & (0\cdot 0 - 4) \\ -(2\cdot 6 - 3\cdot 0) & (6\cdot 1 - 3\cdot 1) & -(1\cdot 0 - 2\cdot 1) \\ (2\cdot 5 - 4\cdot 3) & -(1\cdot 5 - 3\cdot 0) & (1\cdot 4 - 2\cdot 0) \end{bmatrix} = \begin{pmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{22}\begin{pmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{pmatrix} \longrightarrow \frac{1}{|A|}(C_{ij})^T$$

Hermetian $A = A^{+}$ $\quad A^{+} = (A^{T})^{*}$

Operaters in QM MUST!! be hermetian

For Hermitian $\hat{A}^{+}=\hat{A}$

$A=A^{+}$? $\quad \langle \hat{T}^{+}\alpha | \beta\rangle = \langle \alpha | \hat{T}\beta\rangle$ For all $|\alpha\rangle, |\beta\rangle$

$\rightarrow$ Means $\langle \hat{Q} | \hat{A} | \psi\rangle = \langle \hat{A}\phi | \psi\rangle$ Since $\hat{A}^{+}=\hat{A}$

First $\quad \langle \alpha | c\beta\rangle = c\langle \alpha | \beta\rangle$, $\langle c\alpha | \beta\rangle = c^{*}\langle \alpha | \beta\rangle$ ← conjugate side

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Hermitian Operators/ Transformations $\rightarrow$ always in orthonormal basis

eigenvalue $\quad \langle \psi_n | \psi_m\rangle = \delta_{mn}$

1. eigenvalues real $\quad$ remember $\hat{Q}|\psi\rangle = q|\psi\rangle$

operator ↑ eigenfunction ↑

$\lambda \rightarrow$ Eigenvalue of $\hat{Q}$ $\rightarrow \langle \psi | \hat{Q}\psi\rangle = \langle \psi | \lambda\psi\rangle = \lambda\langle \psi | \psi\rangle = 1$

$\langle \hat{Q}\psi | \psi\rangle = \langle \lambda\psi | \psi\rangle = \lambda^{*}\langle \psi | \psi\rangle = \lambda^{*}$

$\langle \psi | \hat{Q} | \psi\rangle = \langle \hat{Q}\psi | \psi\rangle$ So only true if $\lambda=\lambda^{*}$ λ real

2. Eigenvectors of distinct eigenvalues are orthogonal

$\hat{Q}|\psi_n\rangle = \lambda_n|\psi_n\rangle \quad \hat{Q}|\psi_m\rangle = \lambda_m|\psi_n\rangle \quad \lambda_n \neq \lambda_m$

$\langle \psi_m | \hat{Q}\psi_n\rangle = \lambda_n\langle \psi_m | \psi_n\rangle \quad \langle \hat{Q}\psi_m | \psi_n\rangle = \lambda_m\langle \psi_m | \psi_n\rangle$

$\lambda_n\langle \psi_m | \psi_n\rangle = \lambda_m\langle \psi_m | \psi_n\rangle$ iff $\langle \psi_m | \psi_n\rangle = 0$

3.) Eigenvectors Span Space For Hermitian Operator $\hat{Q}$

Set of ψ_n's for complete, orthonormal space

Eigenvectors, Eigenvalues.

$\hat{Q}$ is hermitian, has set of space spanning orthonormal $|\psi_n\rangle$'s

ST $\langle\psi_m|\psi_n\rangle = \delta_{mn}$

For $\hat{Q}$ = analytic type operator Solve DE

how about $\hat{Q} = (\ )$ $\psi_n = ?$ $\lambda_n = ?$

For an nxn Matrix representing a hermetian Operator

There will be n eigenvectors and eigenvalues

Let $\hat{Q}|\psi_n\rangle = \lambda_n|\psi_n\rangle$ $(\hat{Q} - \lambda_n\mathbb{1})|\psi_n\rangle = 0$ by assumption

$|\psi_n\rangle \neq 0$ So $\hat{Q} - \lambda\mathbb{1} = 0$ iff $\mathrm{Det}|\hat{Q} - \lambda\mathbb{1}| = 0$

Order equation = n $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Collection of λ_ns = Spectrum, degenerate if more

Than one λ_n is repeated

degenerate $\hat{Q}|\psi_n\rangle = \hat{Q}|\psi_m\rangle = q_n$ Two or more eigenvectors have same eigenvalue $\rightarrow$ next semester.

Example $\rightarrow$ $\hat{Q} = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$ is $\hat{Q}$ hermetian?

$$\hat{Q}^T = \begin{pmatrix} 2 & -2i & 1 \\ 0 & i & 0 \\ 2 & 2i & -1 \end{pmatrix} \quad (\hat{Q}^T)^* = \begin{pmatrix} 2 & 2i & 1 \\ 0 & i & 0 \\ 2 & -2i & -1 \end{pmatrix} \quad \text{No}$$

Can't represent an operator in QM still

$$\hat{Q} - \lambda \mathbb{1} = \left| \begin{pmatrix} 2-\lambda & -2i & 1 \\ 0 & i-\lambda & 0 \\ 2 & 2i & -1-\lambda \end{pmatrix} \right| = 0$$

$$= -(2-\lambda)(i-\lambda)(1+\lambda) + 2i(2-\lambda)(-1-\lambda) + 1(\lambda-i)2$$

$$= -\lambda^3 + (1+i)\lambda^2 - i\lambda = \lambda\left[\lambda^2 - (1+i)\lambda + i\right] = \lambda\left[(\lambda-i)(\lambda-1)\right]$$

$\boxed{\lambda = 0, 1, i}$ have eigenvalues $\rightarrow$ eigenfunctions?

$$\boxed{\hat{Q}|\psi_n\rangle = \lambda_n|\psi_n\rangle}$$

1. Pick each ?

2. call $\psi_n = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, determine a's

$$\hat{Q}|\psi_0\rangle = 0|\psi_0\rangle \quad \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$2a_1 + 0a_2 - 2a_3 = 0 \quad \rightarrow 2a_1 = +2a_3 \Rightarrow a_1 = +a_3$

$-2ia_1 + ia_2 + 2ia_3 = 0 \quad \rightarrow -2ia_1 + ia_2 + 2ia_1 = 0$

$a_1 + 0a_2 - a_3 = 0 \quad a_2 = 0$

Pick $a_1 = A$ so $a_3 = A$ $a_2 = 0$ $|\psi_0\rangle = A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ normalize

$$\langle\psi_0|\psi_0\rangle = A^2 \overline{(1\ 0\ 1)}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 = A^2(1+1) = 2A^2 \quad A = \frac{1}{\sqrt{2}}$$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\hat{Q}|\psi_1\rangle = 1|\psi_1\rangle \quad \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$\rightarrow 2a_1 - 2a_3 = a_1 \rightarrow a_1 = 2a_3$

$-2ia_1 + ia_2 + 2ia_3 = a_2$

$-2ia_1 + 2ia_3 = a_2(1-i)$

$-4ia_3 + 2ia_3 = a_2(1-i)$

$-2ia_3 = a_2(1-i)$

$a_1 = A$

$$|\psi_1\rangle = \begin{pmatrix} A \\ \frac{iA}{1-i} \\ \frac{A}{2} \end{pmatrix}$$

$a_2(1-i) = -2iA + iA = -iA$

$$a_2 = \frac{-iA}{1-i} = \frac{iA}{i-1}$$

$$\langle\psi_1|\psi_1\rangle = A^2 + \left(\frac{iA}{i-1} \cdot \frac{-iA}{(1-i)}\right) + \frac{A^2}{4} = 1 \quad A^2\left[1 + \frac{1}{4} + \frac{1}{i+1-1+i}\right]$$

$$\hat{Q}|\psi_1\rangle = 1|\psi_1\rangle \rightarrow \begin{pmatrix}2 & 0 & 2\\ -2i & i & 2i\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix} = 1\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}$$

$$2a_1 - 2a_3 = a_1 \quad a_1 = 2a_3 \rightarrow \left[a_3 = \frac{a_1}{2}\right]$$

$$a_1 = A$$

$$-2ia_1 + ia_2 + 2ia_3 = a_2$$

$$-2ia_1 + ia_1 = a_2(1-i)$$

$$a_2 = \frac{-ia_1}{1-i}\frac{(1+i)}{(1+i)} = \left[\frac{(1-i)a_1}{2}\right]$$

$$|\psi_1\rangle = \begin{pmatrix}A\\ (\frac{1-i}{2})A\\ \frac{A}{2}\end{pmatrix}$$

$$\langle\psi_1|\psi_1\rangle = A^2 + \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right)A^2 + \frac{A^2}{4} = A^2\left[1 + 1 + \frac{1}{4}\right] = \frac{9}{4}$$

$$|\psi_1\rangle = \sqrt{\frac{4}{9}}\begin{pmatrix}1\\ \frac{1-i}{2}\\ \frac{1}{2}\end{pmatrix}$$