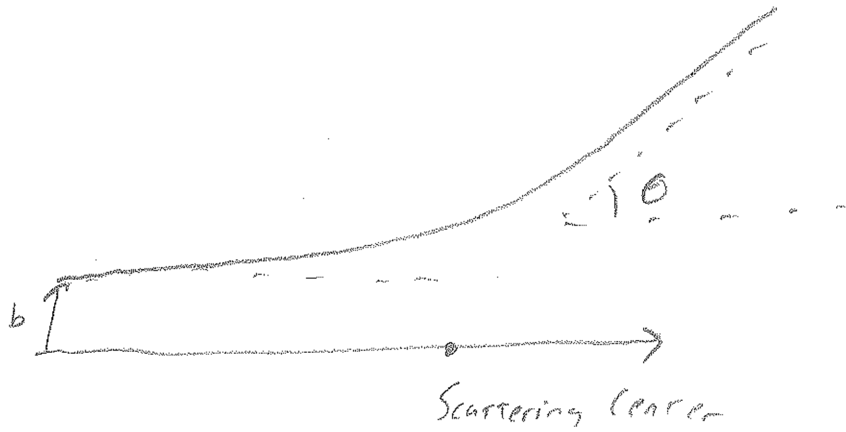
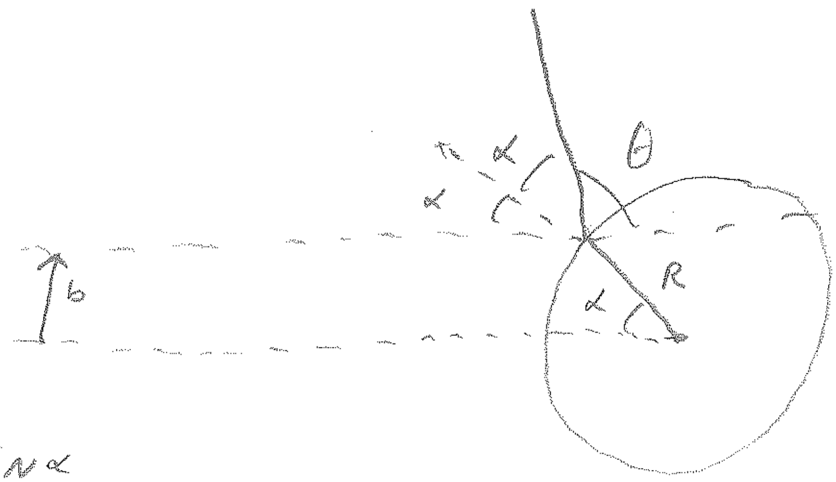


given b get theta



$b = R\sin\alpha$

$b = R\sin\left[\frac{\pi}{2} - \frac{\theta}{2}\right] = R\cos\left(\frac{\theta}{2}\right)$

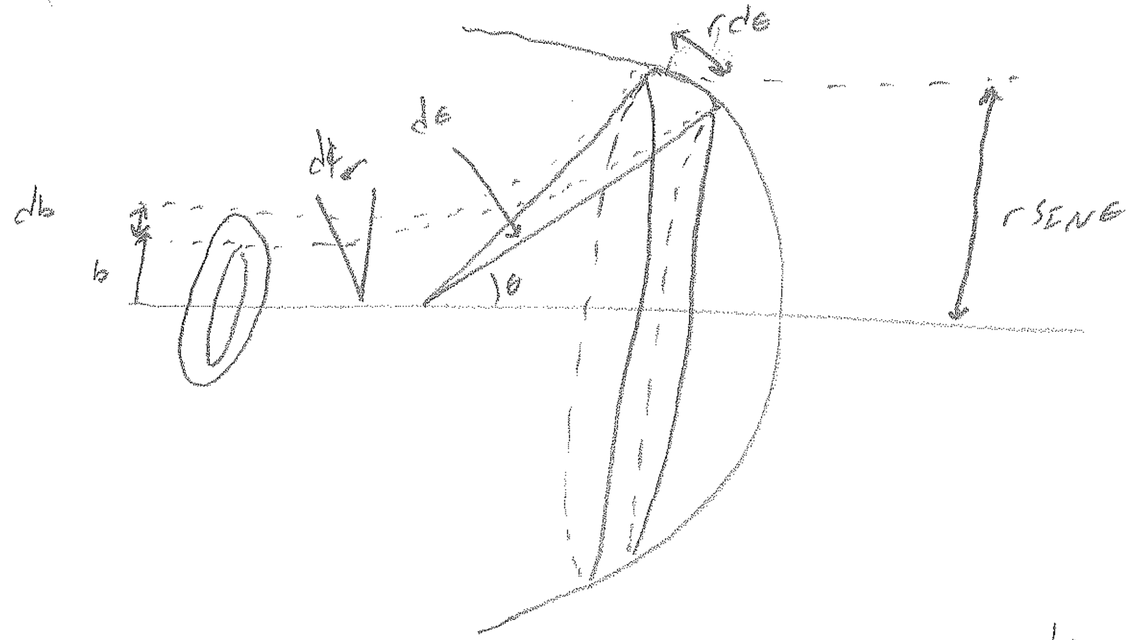
$$\theta = \begin{cases} 2\cos^{-1}(b/R) & \text{if } b \le R \\ 0 & \text{if } b \ge R \end{cases}$$

---

really want to know $d\sigma$ into $d\Omega$

Looking for $D(\theta) = \frac{d\sigma}{d\Omega}$ differential (Scattering) cross-section

$d\sigma = D(\theta)\, d\Omega$ into $d\sigma \rightarrow$ into $d\Omega$



$$d\sigma = b\,db\,d\phi \qquad d\Omega = \sin\theta\,d\theta\,d\phi$$

$$D(\theta) = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For hard Sphere $\frac{db}{d\theta} = -\frac{1}{2} R \sin\left(\frac{\theta}{2}\right)$

$$D(\theta) = \frac{b}{\sin\theta} \cdot \left(\frac{1}{2} R \sin\left(\frac{\theta}{2}\right)\right) = \frac{R\cos\left(\frac{\theta}{2}\right)}{\sin\theta}\left(\frac{1}{2} R \sin\frac{\theta}{2}\right)$$

$$\underline{\underline{D(\theta) = \frac{R^2}{4}}}$$

Total cross section

$$\sigma = \int D(\theta)\, d\Omega$$

Hard Sphere

$$\sigma = \frac{R^2}{4} \int d\Omega = \pi R^2$$ ← cross sectional Sphere

4π

Luminosity

$L =$ # incident per unit area per unit time

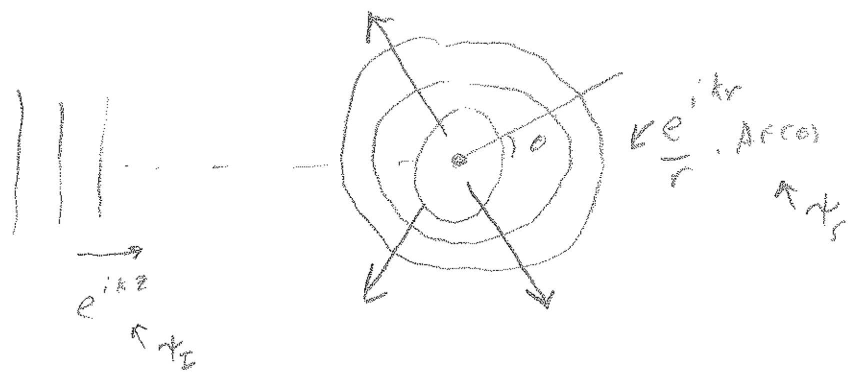
$dN = L\, d\sigma = L\, D(\theta)\, d\Omega$ So

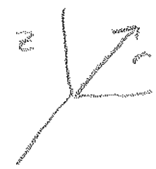
$D(\theta) = \frac{1}{L} \frac{dN}{d\Omega}$ ← count # scattered into

$d\Omega$

Partial Wave Analysis

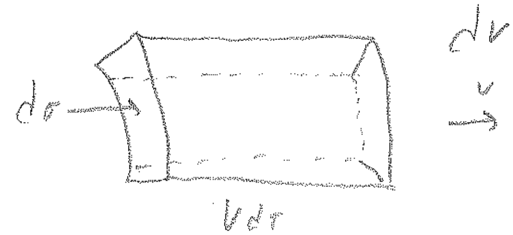
$\psi_I(z) = A e^{ikz}$ where $k = \frac{\sqrt{2mE}}{\hbar}$





$$\psi(r,\theta) = A\left[\underbrace{e^{ikz}}_{\text{plane}} + \underbrace{f(\theta)\frac{e^{ikr}}{r}}_{\text{spherical}}\right] \quad \text{large } r$$

$f(\theta) \rightarrow$ amplitude as function of scattering θ. $f(\theta)$ is what we want



$$\underbrace{dP}_{\text{differential probability of particle passing with velocity } v \text{ through } d\sigma} = \psi_I^2\, dV = \underbrace{|A|^2}_{\psi_I^2}\ \underbrace{v\,dt\,d\sigma}_{dV}$$

but this equals the probability that particles scatters into $d\Omega$

So $dP = |\psi_s|^2 dV = \frac{|A|^2 |f|^2}{r^2} v\,dt\, r^2 d\Omega$

$\uparrow$ volume element scale factor

or $|A|^2 v\, d\sigma = |A|^2 |F|^2 d\Omega$

or $\frac{d\sigma}{d\Omega} = |F|^2$ need scattering amplitude as function of theta

To calculate differential crossection

1D $\psi(x) = A\left[e^{ikx} + F(x/|x|)\, e^{ikx}\right]$

2D $\psi(r,\theta) = A\left[e^{ikr} + F(\theta)\frac{e^{ikr}}{\sqrt{r}}\right]$

Solving for $F(\theta)$ → for $V = V(r)$

$\psi(r,\theta,\phi) = R(r)\, Y_\ell^m(\theta,\phi)$ ↖ spherical harmonic

↗ Find?

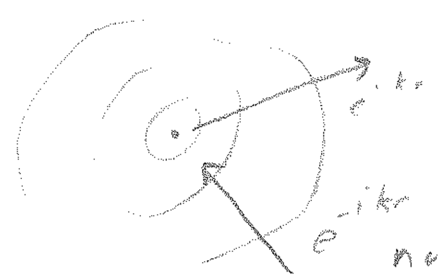
$u(r) = r R(r)$

Then $\frac{-\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{(\ell)(\ell+1)}{r^2}\right]u = Eu$

For $r \to \infty$, assuming $V(r)$ is localized so it goes to zero

For large r $\frac{d^2u}{dr^2} = -k^2 u \to u(r) = Ce^{ikr} + De^{-ikr}$

but



$D = 0$

So $R(r) = \frac{u(r)}{r} = \underline{\frac{Ae^{ikr}}{r}}$ Region 3

True for $kr \gg 1$ or $r \gg \frac{1}{k}$ or $r \gg \frac{\lambda}{2\pi}$

<u>radiation</u> zone look for $r \gg \lambda$ of particle

Where $V \sim 0$ but centrifugal term $\neq 0$

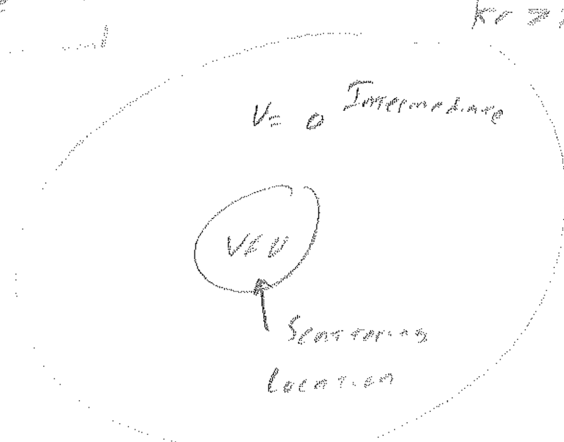
$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u = -k^2 u$$

$$u(r) = \underbrace{A r j_\ell(kr) + B r n_\ell(kr)}_{\text{Spherical Bessel functions}}$$

$j_\ell(kr) \sim$ outgoing, $n_\ell(kr) \sim$ incoming

need $\underbrace{h_\ell^{(1)} = j_\ell(x) + i n_\ell(x) \text{ and } h_\ell^{(2)} = j_\ell - i n_\ell(x)}_{\text{Spherical Hankel functions}}$

radiation zone $kr \gg 1$



$h_\ell^{(1)} \sim \frac{e^{ikr}}{r}$ $\cancel{h_\ell^{(2)} \sim \frac{e^{-ikr}}{r}}$

$$\boxed{\text{gives } R(r) \sim h_\ell^{(1)}(kr)}$$

or

$$\psi(r,\theta,\phi) = A\left[e^{ikz} + \sum_{\ell,m} C_{\ell,m} h_\ell^{(1)}(kr) Y_\ell^m(\theta,\phi)\right] \quad \text{For } V(r)=0$$

For Spherically Symmetric no ϕ dependence so m=0

So $Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$ ← Legendre Polynomial

With $C_\ell^0 = i^{\ell+1} k \sqrt{4\pi(2\ell+1)}\, a_\ell$

and $\psi(r,\theta) = A\left[e^{ikz} + k\sum i^{\ell+1}(2\ell+1)\, a_\ell\, h_\ell^{(1)}(kr) P_\ell(\cos\theta)\right]$

Partial wave amplitude

Hankel function

For $x \gg 1$ $h_\ell^{(1)} \sim \frac{1}{x}(-i)^{\ell+1} e^{ix}$

Or $\psi(r,\theta) \approx A\left[e^{ikz} + F(\theta)\frac{e^{ikr}}{r}\right]$

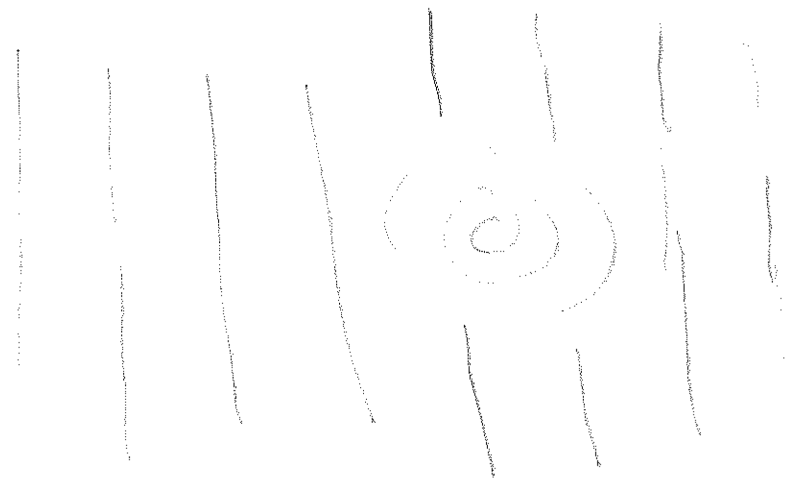
Where $F(\theta) = \sum(2\ell+1)\, a_\ell P_\ell(\cos\theta)$

and $D(\theta) = \frac{d\sigma}{d\Omega} = |F(\theta)|^2 = \sum_\ell\sum_{\ell'}(2\ell+1)(2\ell'+1)\, a_\ell^* a_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$

With $\sigma = 4\pi\sum_{\ell=0}^{\infty}(2\ell+1)|a_\ell|^2$

need a_ℓ

Partial Waves recap



$$\psi_{r \gg} = A\left[e^{ikz} + \frac{e^{ikr}}{r}\right]$$

$\psi(r,\theta)$ in intermediate region $= A\left[e^{ikz} + k\sum_{\ell=0}^{\infty} i^{\ell+1}(2\ell+1)a_\ell h_\ell^{(1)}(kr) P_\ell(\cos\theta)\right]$

$$F(\theta) = \sum (2\ell+1) a_\ell P_\ell(\cos\theta)$$

$$D(\theta) = \frac{d\sigma}{d\Omega} = \sum_\ell \sum_{\ell'} (2\ell+1)(2\ell'+1) a_\ell^* a_{\ell'} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)$$

$$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2$$

Need solution for $V(r) \neq 0$ and match to get $|a_\ell|$

general $V=0$ solution is e^{ikz} or, equivalently

$$\sum_{\ell,m} \left(A_{\ell m} j_\ell(kr) + B_{\ell m} n_\ell(kr)\right) Y_\ell^m(\theta,\phi)$$

but $m=0$ no ϕ dependence and $n_\ell(kr) = \infty$ at $r=0$

So

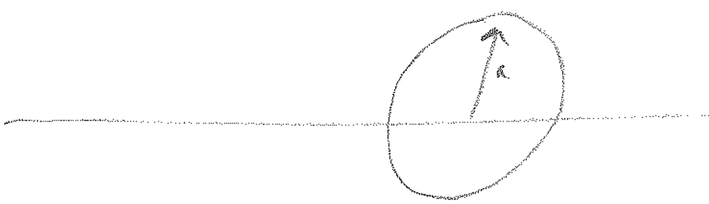
$$e^{ikz} = \sum_\ell A_\ell \, j_\ell(kr) Y_\ell^0(\theta) = \sum i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

$$\boxed{\text{So} \quad \psi(r,\theta) = A \sum_\ell i^\ell (2\ell+1) \left[j_\ell(kr) + ik a_\ell h_\ell^{(1)}(kr) \right] P_\ell(\cos\theta)}$$

USE WITH Matching BC'S

Ex. Let

$$V(r) = \begin{cases} \infty & r \le a \\ 0 & r > a \end{cases}$$



$\psi(a,\theta) = 0$

Or

$$\sum_{\ell=0}^{\infty} i^\ell (2\ell+1) \left[j_\ell(ka) + ik a_\ell h_\ell^{(1)}(ka) \right] P_\ell(\cos\theta) = 0$$

Trick

$$\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta = \frac{2}{(2\ell+1)} \delta_{\ell\ell'}$$

multiply by $P_{\ell'}(\cos\theta)\sin\theta$ then integrate

$$\int \sum i^\ell (2\ell+1)\left[j_\ell(ka) + ika_\ell h_\ell^{(1)}(ka)\right] P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \sin\theta\, d\theta$$

$$= \sum i^\ell (2\ell+1)\left[j_\ell(ka) + ika_\ell h_\ell^{(1)}(ka)\right] \frac{2}{2\ell+1}$$

$$= 2i^\ell \left[j_\ell(ka) + ika_\ell h_\ell^{(1)}(ka)\right] = 0$$

Or $$a_\ell = \frac{i\, j_\ell(ka)}{k\, h_\ell^{(1)}(ka)}$$

and $$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2 = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) \left|\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}\right|^2$$

? hmm Look at $ka = \frac{2\pi a}{\lambda} \ll 1$ or $\lambda \gg a$

Low energy $k \sim \sqrt{E}$ $\lambda \sim \frac{1}{\sqrt{E}}$ $E\downarrow$ $\lambda\uparrow$

$$\frac{j_\ell(z)}{h_\ell^{(1)}(z)} = \frac{j_\ell(z)}{j_\ell(z) + i n_\ell(z)}$$ For large z $n_\ell \gg j_\ell$ Table 4.4

So $\simeq -i\dfrac{j_\ell(z)}{n_\ell(z)}$ using assymptotic Form

$$-i\frac{j_\ell(z)}{n_\ell(z)} = -i\frac{2^\ell \ell!\, z^\ell}{(2\ell+1)!} \cdot \frac{2^\ell \ell!}{-(2\ell)!\, z^{\ell-1}} = \frac{i}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 z^{2\ell+1}$$

So $\sigma = \dfrac{4\pi}{k^2} \sum \dfrac{1}{(2\ell+1)} \left[\dfrac{2^\ell \ell!}{(2\ell)!} \right] (ka)^{4\ell+2}$

For $ka \ll$ $\ell = 0$ dominates

$\sigma \approx \dfrac{4\pi}{k^2} \left[1 \, (1) \, (ka)^2 \right] = 4\pi a^2$

or $4 \times$ Classical cross section

bb vs wave