WKB Approximation

Estimation Technique for TISE

assume Particle energy E $V(x)$ = Constant

if $E > V$ $\psi(x) = Ae^{\pm ikx}$ $k = \sqrt{2m(E-V)}/\hbar$ → wave

+ → − ←

$\lambda = \frac{2\pi}{k}$

Now $V(x)$ not constant but $V(x)$ varies slowly compared to λ

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if $E < V$ $\psi(x) = Ae^{\pm \kappa x}$ $\kappa = \sqrt{2m(V-E)}/\hbar$

If $V(x)$ varies slowly compared with $\frac{1}{\kappa}$ ψ essentially exponential

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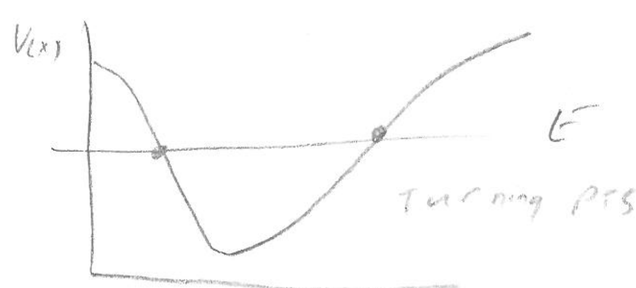
Where $E = V$ difficult



$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi = E\psi$$

or

$$\psi'' = -\frac{p^2}{\hbar^2}\psi \quad \text{where} \quad p = \sqrt{2m(E-V(x))} \quad \text{For } E > V(x) \quad \underline{\underline{\text{real } p}}$$

Let $\psi(x) = A(x)\, e^{i\phi(x)}$

Amplitude ↑ ↖ phase

$$\psi'(x) = A'(x)e^{i\phi(x)} + A\,i\phi'(x)e^{i\phi(x)} = (A' + iA\phi'(x))e^{i\phi(x)}$$

$$\psi''(x) = [A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2]e^{i\phi}$$

or $A'' + 2iA'\phi' + iA\phi'' - A(\phi')^2 = -\frac{p^2}{\hbar^2}A$

Real $A'' - A(\phi')^2 = \frac{-p^2}{\hbar^2}A$ and Imaginary $2A'\phi' + A\phi'' = 0$

$$I \rightarrow 2A'\phi' + A\phi'' = (A^2\phi')' \rightarrow 2AA'\phi' + A^2\phi'' = 0$$

$$(A^2\phi')' = 0 \longrightarrow A^2\phi' = C^2 \rightarrow \boxed{A = \frac{C}{\sqrt{\phi'}}} \quad \text{Need } \phi'$$

Now assume A'' slow compared to rest

$$A'' - A(\phi')^2 = \frac{-p^2}{\hbar^2} A \rightarrow \frac{A''}{A} - (\phi')^2 = -\frac{p^2}{\hbar^2}$$

$$-(\phi')^2 = \frac{-p^2}{\hbar^2} \rightarrow \phi' = \pm \frac{p}{\hbar}$$

$$\frac{d\phi}{dx} = \pm \frac{p(x)}{\hbar} \quad d\phi = \pm \frac{p(x)}{\hbar} dx$$

$$\phi = \pm \frac{1}{\hbar} \int p(x) dx$$

$$\psi(x) \approx A(x) e^{i\phi(x)} \rightarrow \frac{c}{\sqrt{\phi'}} e^{i\phi(x)}$$

$$\psi(x) \cong \frac{c}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx}$$

WKB Continued

$\phi(x) = \pm \frac{1}{\hbar} \int p(x)\,dx$ ← indefinite Integral

$(A^2 \phi')' = 0$

$(\phi')^2 = \frac{p^2}{\hbar^2}$

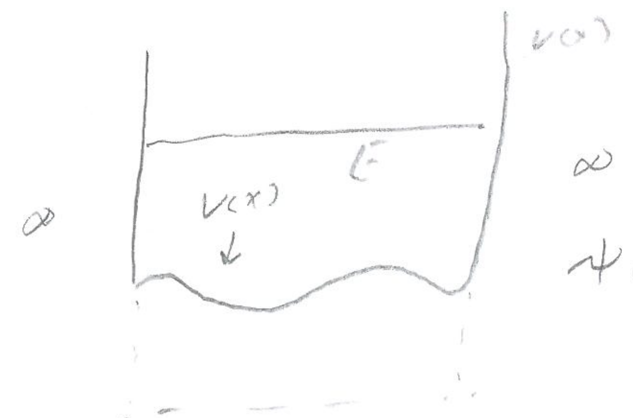
$$\boxed{\psi(x) \cong \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)\,dx}}$$

$|\psi(x)|^2 = \frac{|C|^2}{p(x)}$ $\quad |\psi(x)|^2 dx = \frac{|C|^2 dx}{p(x)}$ → Probability of Finding $\psi(x)$ in $[x, x+dx]$

inversely proportional to it's momentum

$p(x) = \sqrt{2m[E - V(x)]}$ ~ classical momentum

Ex



E > V in well

$\psi(x) \cong \frac{1}{\sqrt{p(x)}} \left[C_+ e^{i\phi(x)} + C_- e^{-i\phi(x)} \right]$

or $\psi(x) = \frac{1}{\sqrt{p(x)}} \left[C_1 \sin[\phi(x)] + C_2 \cos[\phi(x)] \right]$

$\psi(x) = 0$ at $x=0$ or $\phi(0)$ so $C_2 = 0$

$\psi(x) = 0$ at $\phi(a)$ so $\phi(a) = n\pi$

but $\phi(x) = \frac{1}{\hbar} \int_0^x p(x')\,dx'$ or $\phi(a) = \frac{1}{\hbar} \int_0^a p(x')\,dx'$

Or $\int_0^a p(x')dx' = n\pi\hbar$

$\int_0^a p(x)dx = n\pi\hbar$

Integrate

Solve for E

determines approximate energies

For $V(x) = 0$ $\quad p(x) = \sqrt{2mE}$

and $\int_0^a \sqrt{2mE}\, dx' = a\sqrt{2mE} = n\pi\hbar$ or

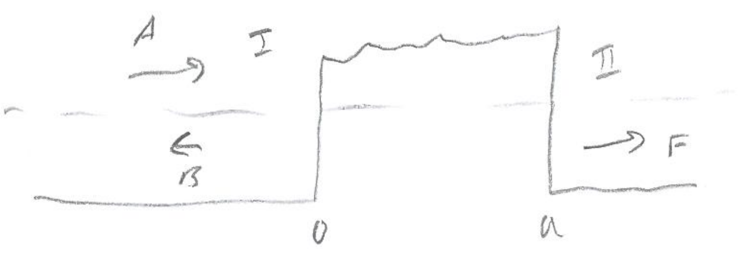
$$E = \left(\frac{n\pi\hbar}{a}\right)^2 \cdot \frac{1}{2m}$$

Tunneling

What if $E < V$

Then $p(x)$ is imaginary and $\psi(x)$ is real

$$\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\pm\frac{1}{\hbar}\int |p(x)|dx}$$



$\psi_I(x) = Ae^{ikx} + Be^{ikx}$

$\psi_{II}(x) = Fe^{ikx}$

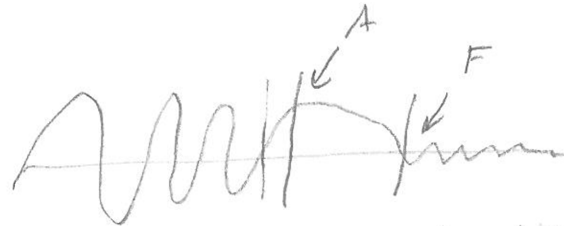
$$T = \frac{|F|^2}{|A|^2}$$

In tunneling region $\psi(x) \cong \frac{C}{\sqrt{|p(x)|}} e^{\frac{1}{\hbar}\int_0^x |p(x')|dx'} + \frac{D}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_0^x |p(x')|dx'}$

For high or wide barrier Probability is Low!

So $C \sim 0$ $C\uparrow$ $T\uparrow$ $D\uparrow$ $T\downarrow$

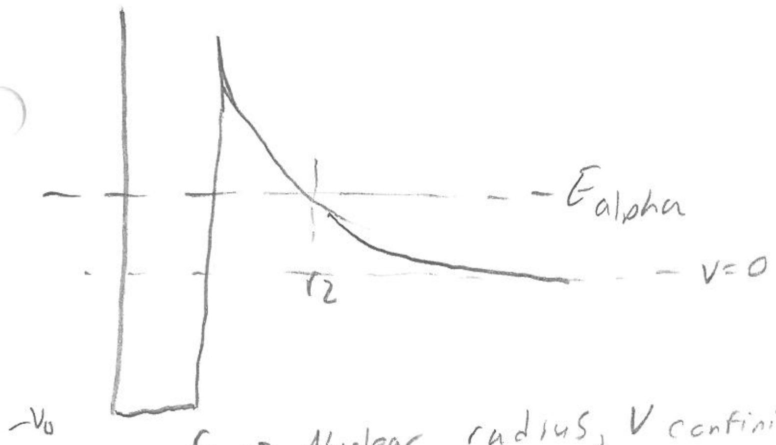
So $\frac{|F|}{|A|} \cong e^{-\frac{1}{\hbar}\int_0^a |p(x')|dx'}$



decrease of Amplitude

$T \cong e^{-2\gamma}$ $\gamma = \frac{1}{\hbar}\int_0^a |p(x')|dx'$

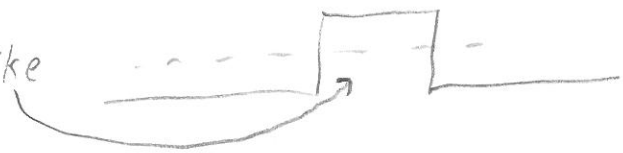
Alpha decay



$r_1 \rightarrow$ Nuclear radius, V confining

$r_2 \rightarrow E$ crosses repulsive coulomb barrier

$[r_1, r_2]$ forbidden regime, Like



Now $E = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r_2}$ Think $\frac{ka}{r}$ $2e \rightarrow$ alpha charge, Ze = nuclear charge

$E = \frac{kq_1q_2}{r}$

Now $\gamma = \frac{1}{\hbar} \int_{r_1}^{r_2} |p(x)|\, dx \qquad |p(x)| = \sqrt{2m(V(r) - E)}$

$V(r) = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r} \qquad |p(r)| = \sqrt{2m\left[\frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r} - \frac{1}{4\pi\epsilon_0}\frac{2Ze^2}{r_2}\right.}$

$V(r) \cdot \frac{r_2}{r_2} = \frac{E r_2}{r} \rightarrow |p(r)| = \sqrt{2m\left(\frac{E r_2}{r} - E\right)}$

$\gamma = \frac{1}{\hbar} \int_{r_1}^{r_2} \sqrt{2mE} \left(\frac{r_2}{r} - 1\right)^{1/2} dr$

$$\gamma = \frac{\sqrt{2mE}}{\hbar} \int_{r_1}^{r_2} \left(\frac{r_2}{r} - 1 \right)^{1/2} dr$$

Let $r = r_2 \sin^2 u$ $\quad dr = 2r_2 \sin(u)\cos(u)\,du$

$$\gamma = \frac{\sqrt{2mE}}{\hbar} \left[r_2 \left(\frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{r_1}{r_2}} \right) \right) - \sqrt{r_1(r_2 - r_1)} \right]$$

For $r_1 \approx 10^{-15}$ $r_2 \approx 10^{-10}$ $r_1 << r_2$ and

$$\boxed{\gamma \cong \sqrt{\frac{2mE}{\hbar^2}} \left[\frac{\pi}{2} r_2 - 2\sqrt{r_1 r_2} \right] = \frac{k_1 Z}{\sqrt{E}} - k_2 \sqrt{Z r_1}}$$

$$k_1 = 1.980\ \text{MeV}^{1/2} = \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{\pi\sqrt{2m}}{\hbar}$$

$$k_2 = 1.485\ \text{Fm}^{-1/2} = \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{4\sqrt{m}}{\hbar}$$

$T \cong e^{-2\gamma}$ $\quad$ Now $\gamma \in [10, 50]$ or e^{-20} to e^{-100} TINY

$\downarrow$
T

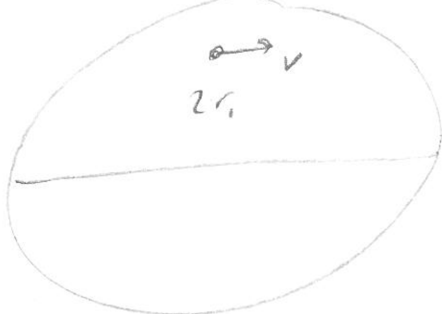
How does tunneling happen?

$e \approx 2.72$ which is between $10^{1/2}$ and $10^{2/5}$

So $T \sim (10^{1/2})^{-[20,100]}$ or $10^{-[10,50]}$

Think of alpha particle as

moving with velocity V in circle of radius r_1



collision time is $\frac{2r_1}{v}$

with frequency $\frac{V}{2r_1} = \nu_c$

multiply tunneling probability $e^{-2\gamma}$ by $\frac{\#\ \text{collisions}}{\text{sec}}$

Probability per second $= T\nu_c = e^{-2\gamma}\frac{V}{2r_1}$

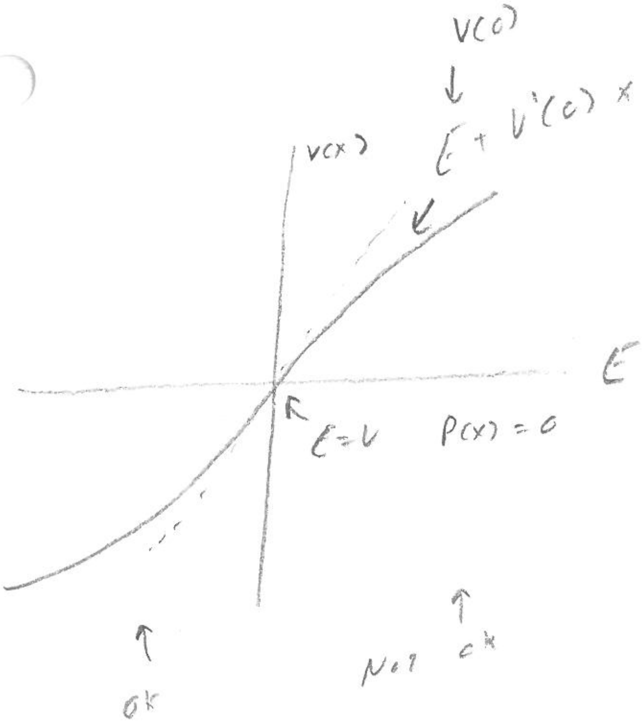
Inverting gives lifetime before collision $= \frac{e^{2\gamma} \cdot 2r_1}{V}$

$V \sim 2 \cdot 10^{7}\ \frac{m}{s}$ $\quad r_1 \sim 10^{-15}\ m$ $\quad \frac{2r_1}{v} \approx \frac{2 \cdot 10^{-15}}{2 \cdot 10^{7}} = 10^{-22}\ s$

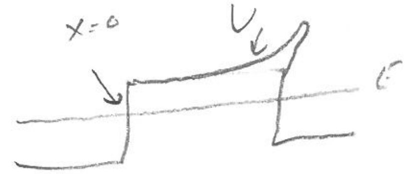
or $e^{2\gamma} \simeq 10^{[10,80]}$ and lifetime $10^{-12}\ s$ to $10^{28}\ s$

Fast to Forever

WKB → connection Formulas



What is ψ

Think 

$$\psi(x) \approx \frac{1}{\sqrt{p(x)}}\left[B e^{\frac{i}{\hbar}\int_x^0 p(x')\,dx'} + C e^{-\frac{i}{\hbar}\int_x^0 p(x')\,dx'}\right] \quad x < 0$$

$$\frac{1}{\sqrt{|p(x)|}}\, D\, e^{-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'} \quad x > 0$$

Assuming $V(x) > E$ for all $x > 0$

Approximate $V(x)$ near $x=0$ as $E + V'(0)\,x$

Solve for a "Patching" ψ near $x=0$

$$-\frac{\hbar^2}{2m}\psi_p'' + [E + V'(0)x]\psi_p = E\psi_p$$

or $\psi_p'' = \frac{2m}{\hbar^2} V'(0)\, x\, \psi_p$

Let $\alpha = \left[\frac{2m}{\hbar^2} V'(0)\right]^{1/3}$ Then $\psi_p'' = \alpha^3 x \psi_p$

Let $z = \alpha x$ Then $\psi_p'' = z \psi_p$ From $\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = \alpha \frac{d}{dz}$, $\frac{d^2}{dx^2} = \alpha^2 \frac{d^2}{dz^2}$

Solutions are Airy Functions

$y'' = zy$ $\quad Ai(z)$, $Bi(z)$ See table 8.1

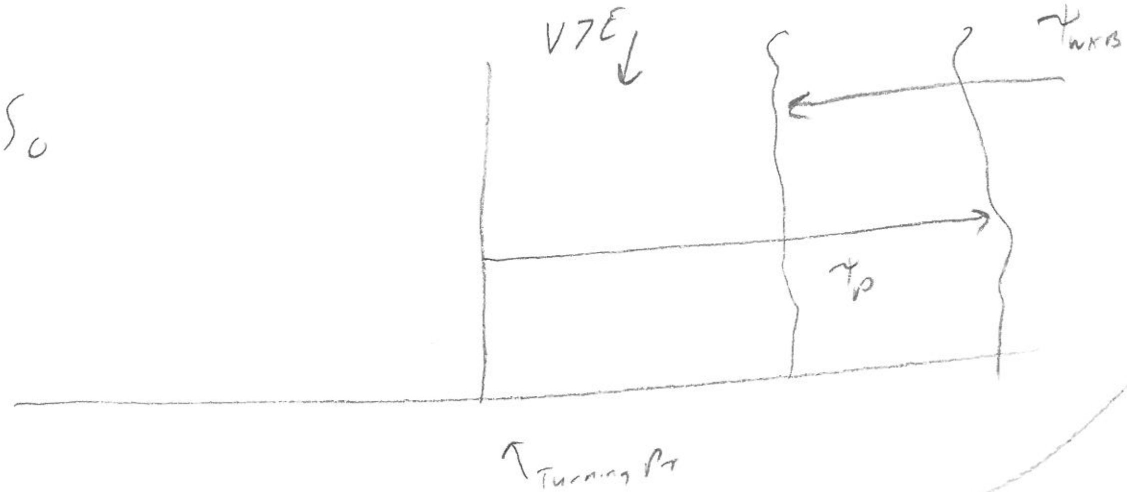
$\psi_p = a\,Ai(x) + b\,Bi(x)$

Near patching region $p(x) \sim \sqrt{2m(E - E - V'(0)x)}$

but $\alpha^3 \frac{\hbar^2}{2m} = V'(0)$ so $p(x) = \alpha^{3/2} \hbar \sqrt{-x}$

and $\int_0^x |p(x')|\,dx' \cong \hbar \alpha^{3/2} \int_0^x \sqrt{x'}\,dx' = \frac{2}{3}\hbar (\alpha x)^{3/2}$

So

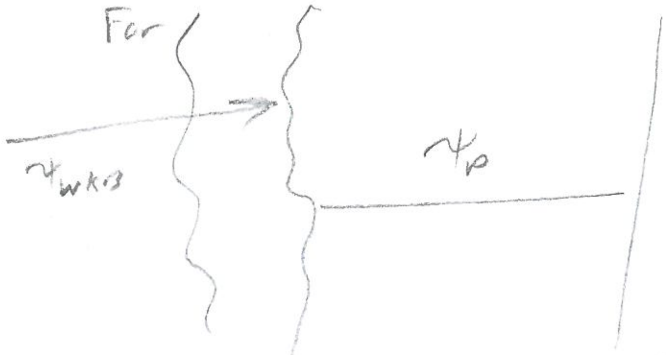


$$\psi(x) \sim \frac{D}{\sqrt{\hbar}\,\alpha^{3/4} x^{1/4}} e^{-\frac{2}{3}(\alpha x)^{3/2}} = a\,Ai(x) + b\,Bi(x)$$

$$= \frac{a}{2\sqrt{\pi}\,(\alpha x)^{1/4}} e^{-\frac{2}{3}(\alpha x)^{3/2}} + \frac{b}{\sqrt{\pi}\,(\alpha x)^{1/4}} e^{\frac{2}{3}(\alpha x)^{3/2}}$$

From which $a = \sqrt{\frac{4\pi}{\alpha\hbar}}\, D \qquad b = 0$

For



$$x < 0 \qquad \int_x^0 |p(x')|\,dx' \simeq \frac{2}{3}\hbar\,(-\alpha x)^{3/2}$$

Now $\psi_p \simeq \dfrac{a}{\sqrt{\pi}\,(-\alpha x)^{1/4}} \sin\left(\frac{2}{3}(-\alpha x)^{3/2} + \frac{\pi}{4}\right)$

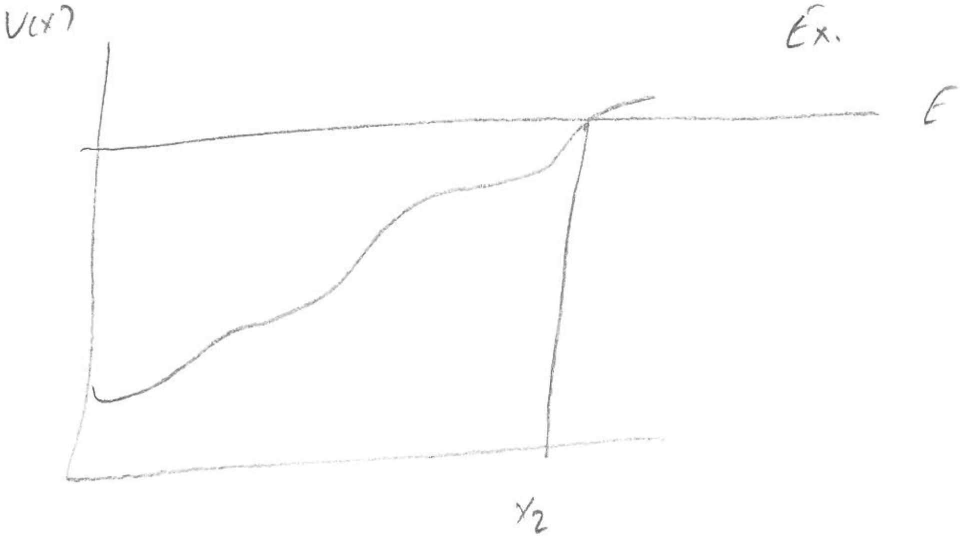
using $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$ and algebra

And comparing to ψ_{WKB} in region 2

$$B = -i e^{i\pi/4} D \qquad C = i e^{-i\pi/4} D$$

$$\psi(x) = \frac{2D}{\sqrt{p(x)}} \sin\left[\frac{1}{\hbar}\int_x^{x_2} p(x')\,dx' + \frac{\pi}{4}\right] \quad \text{For } x < x_2$$

$$\frac{D}{\sqrt{|p(x)|}}\, e^{-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|\,dx'} \qquad x > x_2$$



$$\sin\left[\frac{1}{\hbar}\int_0^{x_2} p(x')dx' + \frac{\pi}{4}\right] = 0 \quad \text{at} \quad x=0$$

$$\text{or} \quad \frac{1}{\hbar}\left[\int_0^{x_2} p(x')\,dx' + \frac{\pi}{4}\right] = n\pi$$

$$\text{or} \quad \int_0^{x_2} p(x')\,dx' = (n-\tfrac{1}{4})\pi\hbar$$

Let $V(x) = \frac{1}{2}m\omega^2 x^2 \quad x>0 \qquad \infty \text{ else}$

$E = \frac{1}{2}m\omega^2 x_2^2$ so $p(x) = \sqrt{2m(E-\frac{1}{2}m\omega^2x^2)}$

$p(x) = m\omega\sqrt{x_2^2 - x^2}$ and $x_2 = \frac{1}{\omega}\sqrt{\frac{2E}{m}}$

$$\int_0^{x_2} p(x)\,dx = m\omega\int_0^{x_2}\sqrt{x_2^2-x^2}\,dx = \frac{\pi}{4}m\omega x_2^2 = \frac{\pi E}{2\omega} = (n-\tfrac{1}{4})\pi\hbar$$

$$E_n = (2n-\tfrac{1}{2})\hbar\omega$$