The Fine Structure of Hydrogen

$$\hat{H}^0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{r}$$

Now $E_1^0 = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]$ $\quad E_n^0 = \frac{E_1^0}{n^2}$

Bohr $\sim \alpha^2 mc^2$ where $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \sim \frac{1}{137}$

Fine Structure $\rightarrow$ due to relativistic corrections and LS coupling

$\sim \alpha^4 mc^2$ or $E^{FS} \sim \left(\frac{1}{137}\right)^2 E_1^0$

Lamb Shift $\sim \alpha^5 mc^2$

hyperfine $\sim \left(\frac{m}{m_p}\right)\alpha^4 mc^2 \rightarrow E^{HF} \sim \frac{1}{1836}\cdot\left(\frac{1}{137}\right)^2 E_1^0$

Small Corrections

just what Perturbation theory is all about

FINE STRUCTURE #1

$\hat{H}^0 = \hat{T} + \hat{V}$ where $\hat{T} \sim \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$

$\hat{T}$ really $(\gamma - 1)mc^2$ where $\gamma = \dfrac{1}{\sqrt{1-(\frac{v}{c})^2}}$

motional Energy

$$\hat{T} = \underbrace{\frac{mc^2}{\sqrt{1-(\frac{v}{c})^2}}}_{\text{KE + Rest Mass}} - \underbrace{mc^2}_{\text{rest mass}} \qquad \text{From } E^2 = p^2c^2 + m^2c^4$$

$$E = \gamma mc^2$$

$$p = \gamma m v = \frac{mv}{\sqrt{1-(\frac{v}{c})^2}}, \quad p^2c^2 + m^2c^4 = \frac{m^2v^2c^2}{1-\frac{v^2}{c^2}} + m^2c^4$$

$$p^2c^2 + m^2c^4 = \frac{m^2v^2c^2 + m^2c^4\left[1-\frac{v^2}{c^2}\right]}{1-\frac{v^2}{c^2}} = \frac{m^2v^2c^2 + m^2c^4 - m^2v^2c^2}{1-\frac{v^2}{c^2}}$$

$$p^2c^2 + m^2c^4 = \frac{m^2c^4}{1-\frac{v^2}{c^2}} = (T + mc^2)^2$$

So $T = \sqrt{p^2c^2 + m^2c^4} - mc^2$

or $T = mc^2\left[\sqrt{1+\left(\frac{p}{mc}\right)^2} - 1\right]$

or $T = mc^2\left[1 + \frac{1}{2}\left(\frac{p}{mc}\right)^2 - \frac{1}{8}\left(\frac{p}{mc}\right)^4 + \cdots - 1\right]$

non-relativistic $\frac{1}{2}\frac{p^2}{m}$ 1st order correction $\frac{-p^4}{8m^3c^2}$

$$\boxed{\text{So } \underset{\uparrow \text{relativistic}}{E_r^1} = \langle H_r^1 \rangle = \frac{-1}{8m^3c^2} \langle \psi | p^4 | \psi \rangle = -\frac{1}{8m^3c^2} \langle p^2\psi | p^2\psi \rangle}$$

But For $\hat{H}^0$ $\hat{p}^2\psi = 2m(E-\hat{V})\psi$ $\hat{H}^0 = \frac{\hat{p}^2}{2m}\psi + \hat{V}\psi = E\psi$

$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$ So $p^4\psi = 4m^2(E-V)^2\psi$

or $-\frac{1}{8m^3c^2} \langle \hat{p}^2\psi | \hat{p}^2\psi \rangle = = -\frac{1}{2mc^2} \langle \psi | (E-\hat{V})^2 | \psi \rangle$

$= -\frac{1}{2mc^2} \langle \psi | E^2 - 2E\hat{V} + \hat{V}^2 | \psi \rangle$

$= -\frac{1}{2mc^2} \left[E^2 - 2E\langle \hat{V} \rangle + \langle \hat{V}^2 \rangle \right]$ $\langle \psi | V^2 | \psi \rangle = \langle V^2 \rangle$

$\langle \psi | E^2 | \psi \rangle = E^2$ $\langle \psi | EV | \psi \rangle = E\langle V \rangle$

Evidently

$$E_r^1 = -\frac{1}{2mc^2}\left[E_n^2 + 2E_n \left(\frac{e^2}{4\pi\epsilon_0}\right) \left\langle \frac{1}{r} \right\rangle + \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \left\langle \frac{1}{r^2} \right\rangle \right]$$

$\left\langle \frac{1}{r} \right\rangle = \frac{1}{n^2 a}$ $\left\langle \frac{1}{r^2} \right\rangle = \frac{1}{(\ell + 1/2)n^3a^2}$

$\langle V \rangle = 2E_n$
4.191

$$E_r^1 = -\frac{(E_n)^2}{2mc^2}\left[\frac{4n}{l+1/2} - 3\right]$$ using 4.72, 4.7 to kill a and express in terms of E_n