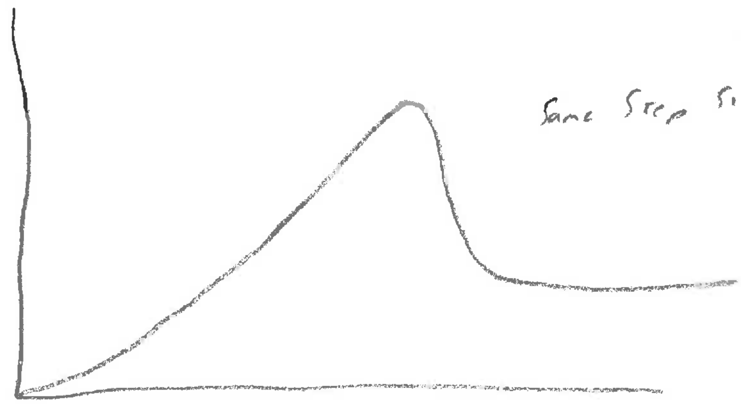
Variable Step Size RK4



Same Step size?

Let h change

1. Pick error allowed (ϵ)

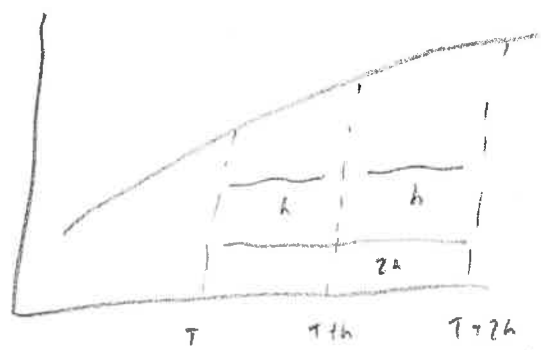
   Ex $\epsilon = .001$ per $T = 1$

   Then $\epsilon_{total} = .01$ for $T \in [0, 10]$

2. compare error to ϵ
3. Change h

Pick an h, small, For 1yr orbit $h \sim 1$ hour

1st



do: 2 individual steps with h 4th order accurate error is 5th order or $\sim Ch^5$ where C is a constant we don't know

Then $X(T+2h) = x_1 + 2Ch^5$

First estimate

do: 1 Step with Step Size 2h

Then $x(t+2h) = x_2 + c(2h)^5 = x_2 + 32ch^5$

x_2 ← Second estimate; $32ch^5$ ← error

Now $x_1 - x_2 = 30ch^5$

Now $\epsilon = ch^5 = \dfrac{(x_1 - x_2)}{30}$

desired $\longrightarrow$ $\epsilon' = ch'^5$

h' ← Target h

$$\epsilon' = ch'^5 = ch^5\left(\frac{h'}{h}\right)^5 = \frac{1}{30}(x_1 - x_2)\left(\frac{h'}{h}\right)^5$$

Pick a target accuracy δ s.t.

$$\delta = \frac{\frac{1}{30}|x_1 - x_2|\left(\frac{h'}{h}\right)^5}{h'} \quad \text{error per desired timestep}$$

δ ← you pick

Then $h' = h\left(\dfrac{30h\delta}{|x_1 - x_2|}\right)^{1/4} = h\rho^{1/4}$

$$\rho = \frac{30(h\delta)}{|x_1 - x_2|} \quad \text{ratio} \quad \frac{(h\delta) \leftarrow \text{target}}{\text{actual accuracy}}$$

$h\delta$ ← target accuracy

Method.
1: Take 2 successive RK4 steps to calculate x_1
2: Take 1 step at 2h to calculate x_2
3: calculate ρ
   a: if $\rho > 1$ fine, calculate new $h' = h\rho^{1/4}$ for next timestep
      move to $x(t+2h)$ with larger step
   b: if $\rho < 1$ bad, not accurate enough
      calculate $h' = h\rho^{1/4}$, reduce step, try again

So either accept, move to $X(T+2h)$, increase h
or reduce h, keep trying

Cons → 3 RK4 steps per implementation
Pros → usually faster, better accuracy

really need a safety factor
if h increases limit the increase by maximum
of $2h$

2D? $\epsilon_x = \frac{1}{30}(x_1 - x_2)$ $\quad \epsilon_y = \frac{1}{30}(y_1 - y_2)$

$\frac{1}{30}|\vec{x}_1 - \vec{x}_2| \longrightarrow \sqrt{\epsilon_x^2 + \epsilon_y^2}$

Making RK4 order h^6?

recall 2 step

$X(T+2h) = x_1 + 2ch^5 + O(h^6)$

$ch^5 = \frac{1}{30}(x_1 - x_2)$

So $x(T+2h) = x_1 + \frac{1}{15}(x_1 - x_2) + O(h^6)$

1 → do above with caveats that $\rho > 1$

Then $x(T+2h) = x_1 + \frac{1}{15}(x_1 - x_2)$

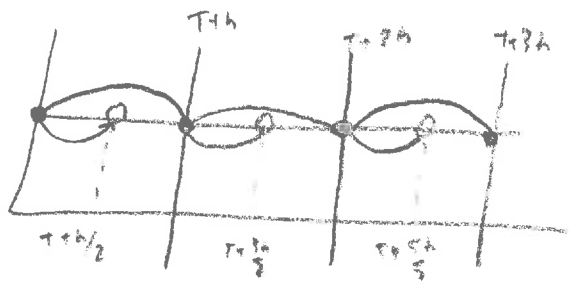
I could make this more annoying

6th order, flexible adaptive, Stiff Solver etc.

just use odeint

For energy and momentum conserving problems it gets easier

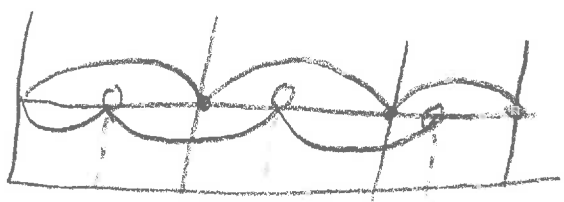
RK4



use midpoint to calculate endpoint

midpoint calculated at beginning of each

Leapfrog



midpoint calculated from midpoint

$$x(T+\tfrac{1}{2}h) = x(T) + \tfrac{1}{2}hF(x,T) \qquad F(x,T) = \frac{dx}{dT}$$

Euler

$$x(T+h) = x(T) + hF(T+\tfrac{1}{2}h,\ x(T+\tfrac{1}{2}h)) + O(h^2)$$

$$x(T+\tfrac{3}{2}h) = x(T+\tfrac{1}{2}h) + hF(T+h,\ x(T+h))$$

$$x(T+2h) = x(T+h) + hF(T+\tfrac{3}{2}h,\ x(T+\tfrac{3}{2}h))$$

$$\vec{r}(T+h) = \vec{r}(T) + hF(T+\tfrac{1}{2}h,\ \vec{r}(T+\tfrac{1}{2}h))$$

$$\vec{r}(T+\tfrac{3}{2}h) = \vec{r}(T+\tfrac{1}{2}h) + hF(T+h,\ \vec{r}(T+h))$$

Why? Time reversible, conserves energy and momentum

Verlet Method

Verlet Method

$\vec{F} = m\ddot{\vec{a}}$ $\quad \frac{dx}{dT} = v \quad \frac{dv}{dT} = F(x,T)$

let $\vec{r} = (x, v)$

$$\frac{d\vec{r}}{dT} = F(r, T)$$

know $x(T)$, $v(T+\frac{1}{2}h)$

$$x(T+h) = x(T) + hv(T+\tfrac{1}{2}h)$$

$$v(T+\tfrac{3}{2}h) = v(T+\tfrac{1}{2}h) + hF(T+h, x(T+h))$$

iterate

works for $x(v)$, $v(x)$ → gravity, Springs

hmm what if we want $E = PE + KE$?

need $v(T+nh)$ not $v(T+\frac{n}{2}h)$

Single backward Euler step

$$v(T+\tfrac{1}{2}h) = v(T+h) - \tfrac{1}{2}hF(T+h, x(T+h))$$

$$\boxed{v(T+h) = v(T+\tfrac{1}{2}h) + \tfrac{1}{2}hF(T+h, x(T+h))}$$

given $x(T)$, $v(T)$

$$\frac{d^2\vec{r}}{dT^2} = F(\vec{r}, T) \rightarrow \overbrace{v(T+\tfrac{1}{2}h) = v(T) + \tfrac{1}{2}hF(T, r(T))}^{\text{once}}$$

iterate

$$r(T+h) = r(T) + hv(T+\tfrac{1}{2}h)$$

$$k = hF(T+h, r(T+h))$$

$$v(T+h) = v(T+\tfrac{1}{2}h) + \tfrac{1}{2}k$$

$$v(T+\tfrac{3}{2}h) = v(T+\tfrac{1}{2}h) + k$$

Why? Time reversal and energy conservation.

lets let $h \to -h$

$$x(T-h) = x(T) - h\,F(T-\tfrac{1}{2}h,\ x(T-\tfrac{1}{2}h))$$

$$x(T-\tfrac{3}{2}h) = x(T-\tfrac{1}{2}h) - h\,F(T-h,\ x(T-h))$$

$T \to T+\frac{3}{2}h$

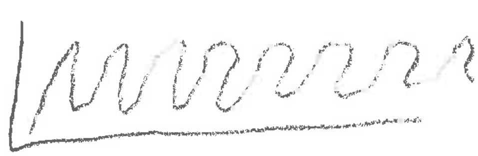
$$x(T+\tfrac{1}{2}h) = x(T+\tfrac{3}{2}h) - h\,F(T+h,\ x(T+h))$$

$$x(T) = x(T+h) - h\,F(T+\tfrac{1}{2}h,\ x(T+\tfrac{1}{2}h))$$

which is exactly the same as going forward with $h \to -h$ so you can start at the end and "retrace" the solution back to the start. This is NOT the case for replacing $h \to -h$ in a RK scheme.

Ex $\dfrac{d\theta}{dT} = \omega \qquad \dfrac{d\omega}{dT} = -\dfrac{g}{\ell}\sin\theta$

E  RK2 energy builds (or decays)

E  Leapfrog

good for periodic systems springs, orbits, etc.