again $\dfrac{\partial T}{\partial t} - \alpha \dfrac{\partial^2 T}{\partial x^2} = \underbrace{\left| \dfrac{T_j^{n+1} - T_j^n}{\Delta t} - \dfrac{\alpha (T_{j+1}^n - 2T_j^n + T_{j-1}^n)}{(\Delta x)^2} \right|}_{\text{difference equation}} +$

$$\underbrace{\left[-\left(\frac{\partial^2 T}{\partial t^2}\right)_j^n \frac{\Delta t}{2} + \alpha \left(\frac{\partial^4 T}{\partial x^4} \frac{(\Delta x)^2}{12} \right) + \cdots \right]}_{\text{truncation error}}$$

Can we do better? Segue

Classes of PDEs and impacts on computation

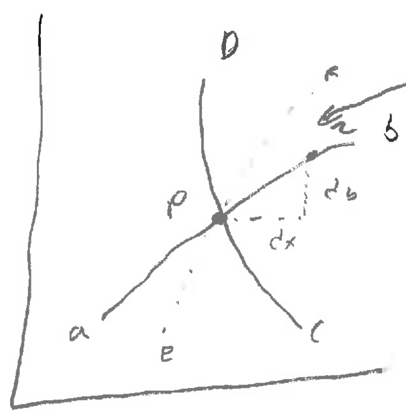
Quasi linear PDES

System $a_1 \dfrac{\partial u}{\partial x} + b_1 \dfrac{\partial u}{\partial y} + c_1 \dfrac{\partial v}{\partial x} + d_1 \dfrac{\partial v}{\partial y} = f_1(x, y, u, v)$

$a_2 \dfrac{\partial u}{\partial x} + b_2 \dfrac{\partial u}{\partial x} + c_2 \dfrac{\partial v}{\partial x} + d_2 \dfrac{\partial v}{\partial y} = f_2(x, y, u, v)$

$a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ may be functions of x, y, u, and v

at any point in xy u, v take on unique values and their first and second derivatives are finite.



Characteristic curve

Lines where $\dfrac{\partial u}{\partial x \text{ or } dy}$, $\dfrac{\partial v}{\partial x \text{ or } dy}$ are indeterminate and may be discontinuous,

WHY? You just said . . . .

Now $du = \dfrac{\partial u}{\partial x} dx + \dfrac{\partial u}{\partial y} dy$ $\quad dv = \dfrac{\partial v}{\partial x} dx + \dfrac{\partial v}{\partial y} dy$

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ du \\ dv \end{bmatrix}$$

How do we get our unique $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$?

$[A]x = b$

$\frac{\partial u}{\partial x}$ 1st

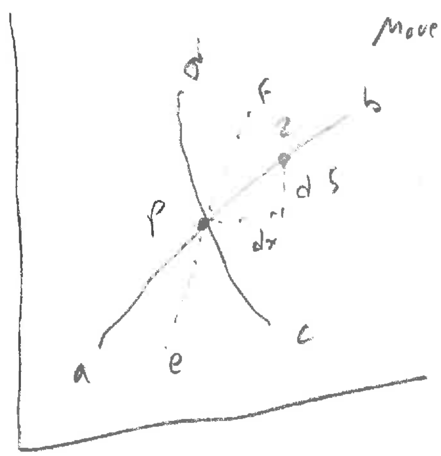
$$[A] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \qquad [B] = \begin{bmatrix} F_1 & b_1 & c_1 & d_1 \\ F_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{bmatrix}$$

Cramers Rule

$$\frac{\partial u}{\partial x} = \frac{|B|}{|A|}$$

$$X_i = \frac{\det[A_i]}{\det[A]}$$

D_i = matrix formed by replacing i-th column of A with vector b



Move along a-b from p-2

$dx = x_2 - x_p \quad dy = y_2 - y_p$

$du = u_2 - u_p \quad dv = v_2 - v_p$

$\frac{\partial u}{\partial x} \to$ unique value

do the same along c-d

Same value has to be, direction shouldn't matter

What if we move along ef and $\det[A] = 0$

So $\frac{\partial u}{\partial x}, \ldots$ don't exist? These are characteristic curves. Lines are determinate $|A| = 0$

What are they?

Solve
$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0$$

1 $(a_1 c_2 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1) dx\, dy + (b_1 d_2 - b_2 d_1)(dx)^2 = 0$

2 divide $\frac{1}{dx^2}$ $(a_1 c_2 - a_2 c_1)\left(\frac{dy}{dx}\right)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\frac{dy}{dx} + (b_1 d_2 - b_2 d_1) = 0$

3. Let $a = (a_1 c_2 - a_2 c_1)$ $b = -(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)$ $c = (b_1 d_2 - b_2 d_1)$

Now $a\left(\frac{dy}{dx}\right)^2 + b\left(\frac{dy}{dx}\right) + c = 0$

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad D = b^2 - 4ac$$

System is

if $D > 0$ 2 real characteristic curves hyperbolic

$D = 0$ 0 characteristics parabolic

$D < 0$ 2 imaginary characteristics elliptic

Comes from analytic geometry $ax^2 + bxy + cy^2 + dx + ey + f = 0$

$b^2 - 4ac > 0$ Conic is a hyperbola

$b^2 - 4ac = 0$ conic is a parabola

$b^2 - 4ac < 0$ Conic is an ellipse

Now if only $|A| = 0$ $\frac{\partial u}{\partial v} \to \infty$ No No

want indeterminate set $|B| = 0$ also

Expansion leads to an ordinary differential equation

which only holds along characteristic curve

$|B| = 0 \rightarrow$ compatibility equation 1D along characteristics. Widely used for inviscid, supersonic flows

---

'much easier if $F_1, F_2 = 0$

Then $w = \begin{Bmatrix} u \\ v \end{Bmatrix}$ $\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial x} \end{pmatrix} + \begin{pmatrix} b_1 & d_1 \\ b_2 & d_2 \end{pmatrix}\begin{pmatrix} \frac{\partial w}{\partial y} \end{pmatrix} = 0$

$$[K]\frac{\partial w}{\partial x} + M\frac{\partial w}{\partial y} = 0$$

$$\frac{\partial w}{\partial x} + \underbrace{[K]^{-1}[M]}_{[N]}\frac{\partial w}{\partial y} = 0$$

real eigenvalues for $[N] \rightarrow$ hyperbolic

complex $\rightarrow$ elliptic

Ex $\quad (1-M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$

$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} = 0$$

$$\underbrace{\begin{bmatrix} 1-M_\infty^2 & 0 \\ 0 & -1 \end{bmatrix}}_{[K]}\frac{\partial w}{\partial x} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{[M]}\frac{\partial w}{\partial y} = 0$$

$$[N] = \begin{bmatrix} 0 & \frac{1}{1-M_\infty^2} \\ -1 & 0 \end{bmatrix} \qquad \lambda = \pm\sqrt{\frac{1}{M_\infty^2 - 1}}$$

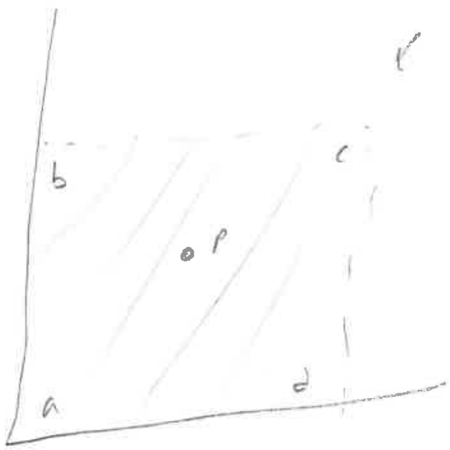
$M_\infty > 1$ hyperbolic $\quad M_\infty < 1$ elliptic

Elliptic $\nabla^2 \phi = 0$ imaginary characteristics

No regions of influence or domains of dependence

Pure boundary value problems.

generally Steady State Solutions



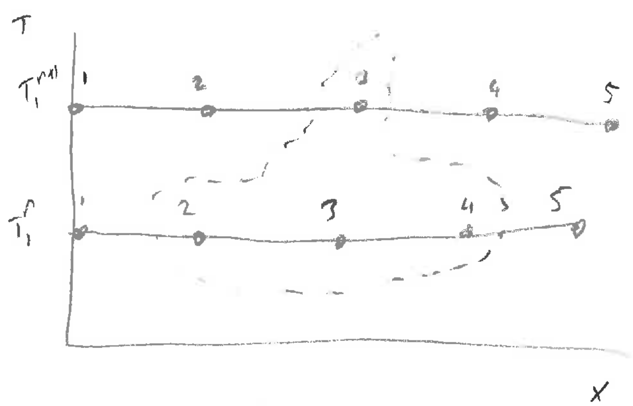
P immediately influences whole region

NOT good for marching

We'll solve this later

Back to heat Conduction Implicit!

Explicit



Time marching

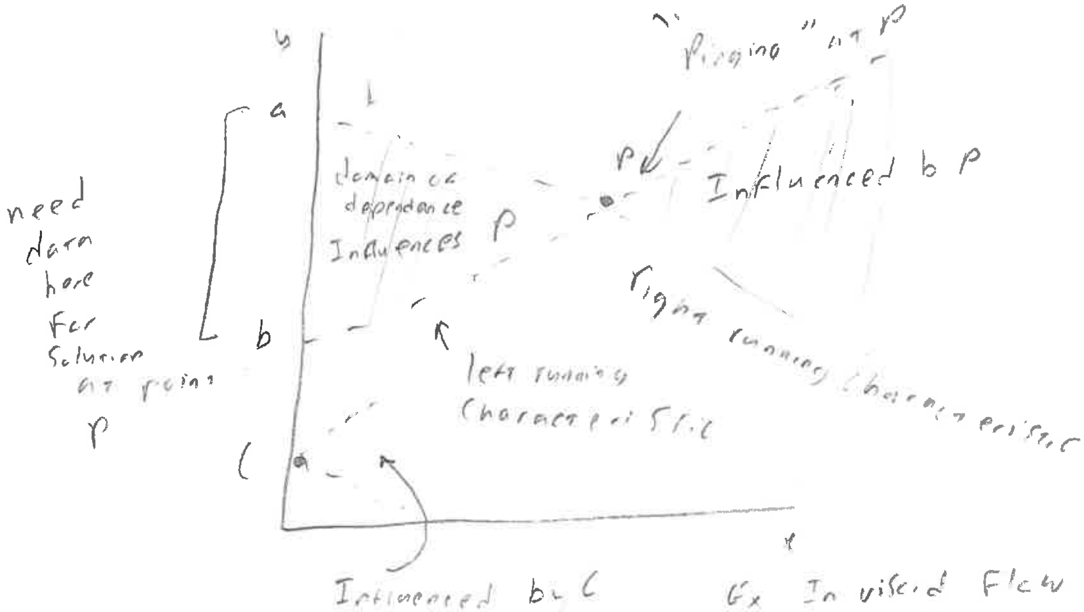
$$T_i^{n+1} = T_i^n + \frac{\alpha \Delta T}{(\Delta x)^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$

$$T_3^{n+1} = T_3^n + \frac{\alpha \Delta T}{(\Delta x)^2}\left(T_4^n - 2T_3^n + T_2^n\right)$$

Solve explicitly 1 unknown time

So what? Affect what and how Solution can be known

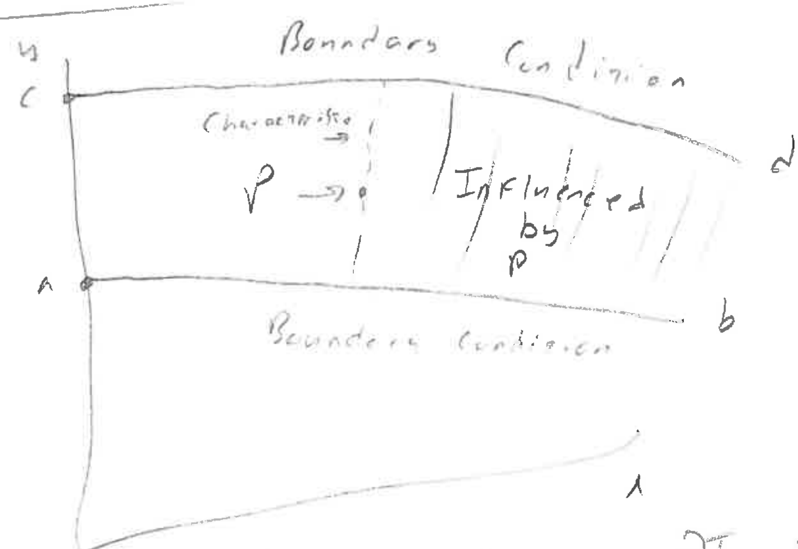
Hyperbolic 2 characteristics



Ex Inviscid Flow

Good for time marching

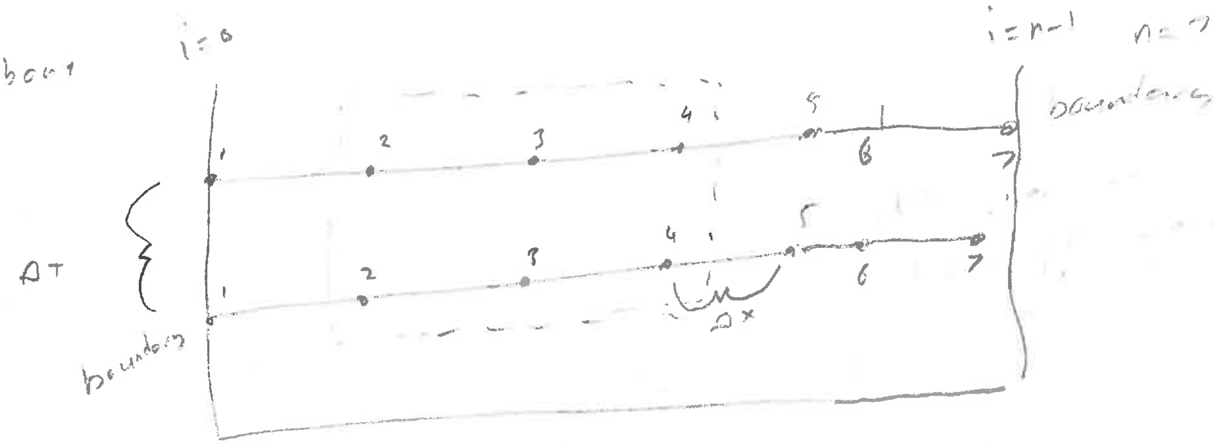
Parabolic 1 characteristic



good for time marching

Ex heat Conduction $\frac{\partial T}{\partial t} + \alpha\frac{\partial^2 T}{\partial x^2}$

How about



$$\frac{T_i^{n+1} - T_i^n}{\Delta T} = \alpha \frac{\frac{1}{2}[T_{i+1}^{n+1} + T_{i+1}^n] + \frac{1}{2}[-2T_i^{n+1} - 2T_i^n] + \frac{1}{2}[T_{i-1}^{n+1} + T_{i-1}^n]}{(\Delta x)^2}$$

Crank-Nicolson

good for Parabolic equations

rearrange

$$\boxed{\frac{\alpha \Delta T}{2(\Delta x)^2} T_{i-1}^{n+1} - \left[1 + \frac{\alpha \Delta T}{(\Delta x)^2}\right] T_i^{n+1} + \frac{\alpha \Delta T}{2(\Delta x)^2} T_{i+1}^{n+1} = -T_i^n - \frac{\alpha \Delta T}{2(\Delta x)^2}\left[T_{i+1}^n - 2T_i^n + T_{i-1}^n\right]}$$

Let $A = \frac{\alpha \Delta T}{2(\Delta x)^2}$ $\quad B = \left(1 + \frac{\alpha \Delta T}{(\Delta x)^2}\right)$ $\quad k_i = -T_i^n - \frac{\alpha \Delta T}{2(\Delta x)^2}\left[T_{i+1}^n - 2T_i^n + T_{i-1}^n\right]$

Then $A T_{i-1}^{n+1} - B T_i^{n+1} + A T_{i+1}^{n+1} = k_i$

At grid point 2 $\quad \underbrace{A T_1 - B T_2 + A T_3}_{\text{not known}} = \underset{\text{known}}{k_2}$ dropped superscript

but T_1 is known → boundary condition

So $-BT_2 + AT_3 = \underbrace{k_2 - AT_1}_{k_2'}$

$-BT_2 + AT_3 = k_2'$

At grid point 3: $AT_2 - BT_3 + AT_4 = k_3$

At grid point 4: $AT_3 - BT_4 + AT_5 = k_4$

At grid point 5: $AT_4 - BT_5 + AT_6 = k_5$

At grid point 6: $AT_5 - BT_6 + AT_7 = k_6$

but T_7 is known so $AT_5 - BT_6 = k_6 - AT_7 = k_6'$

Matrix Equation

$$\begin{bmatrix} -B & A & 0 & 0 & 0 \\ A & -B & A & 0 & 0 \\ 0 & A & -B & A & 0 \\ 0 & 0 & A & -B & A \\ 0 & 0 & 0 & A & -B \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} k_2' \\ k_3 \\ k_4 \\ k_5 \\ k_6' \end{bmatrix}$$

Seems more complicated

why use?

$\Delta T \rightarrow$ can be large

implicit is

unconditionally stable

Tridiagonal

Thomas' algorithm

Generally, implicit is only good for steady state

Otherwise if time marching solution is to be correct

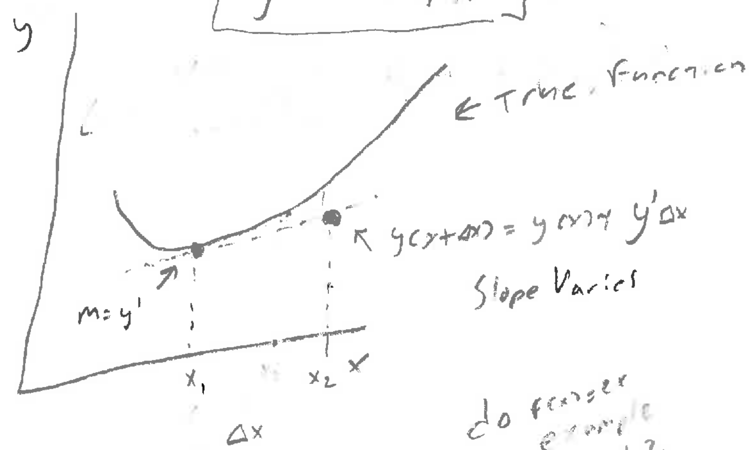
Truncation error dominates

Back To ODEs

$$\frac{dx}{dT} = \frac{(x(T+\Delta T) - x(T))}{\Delta T} = F(T)$$ derivative constant across ΔT?

No

$$\boxed{y' = F(x_i, y_i)}$$



do renser example $x(t)^2$

$$y' = \frac{y(x+\Delta x) - y(x)}{\Delta x} = F(x,y)$$

$$y(x+\Delta x) = y(x) + F(x,y)\Delta x$$

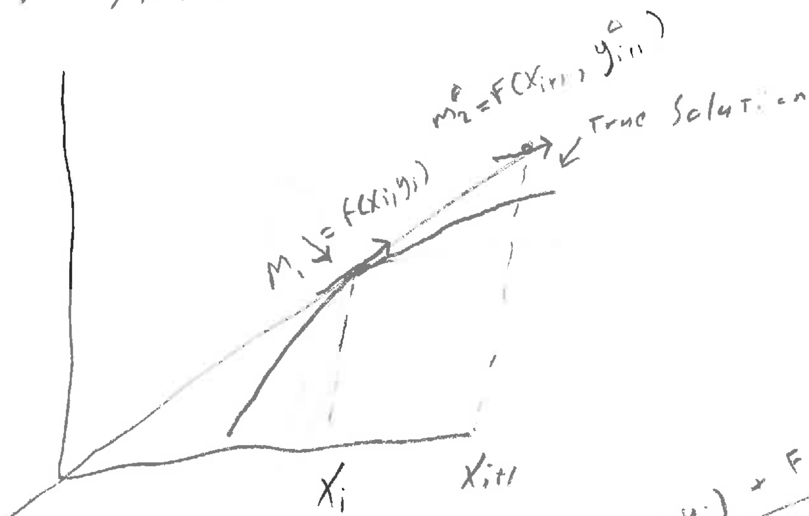
remember $y(x) = \int_0^x F(x)\,dx$

Integrator time

Heun's Method

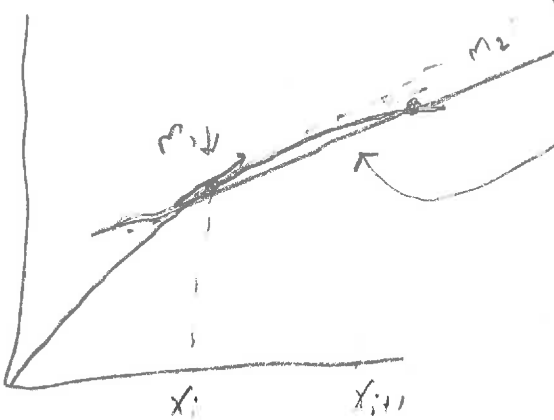
Predictor corrector

Predictor



$$\frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^0)}{2} = \frac{m_1 + m_2}{2}$$

Corrector



Now

Predictor $Y_{i+1}^{0} = y_i + F(x_i, y_i)\Delta x$

Corrector $Y_{i+1} = y_i + \left(\frac{F(x_i, y_i) + F(x_{i+1}, y_{i+1}^{c})}{2}\right)\Delta x$

---

just x $\quad Y_{i+1} = y_i + \left(F(x_i) + F(x_{i+1})\right)\frac{h}{2}$ midpoint method

---

Runge Kutta Schemes → Integration schemes

$y_{i+1} = y_i + \phi(x_i, y_i, h)h$

$\phi = a_1 k_1 + a_2 k_2 + \dots a_n k_n$ → increment function

$k_1 = F(x_i, y_i)$

$k_2 = F(x_i + p_2 h, y_i + q_{11} k_1 h)$

$k_3 = F(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$

$k_n = F(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \dots q_{n-1,n-1} k_{n-1} h)$

essentially a multislope evaluation with weights.

2nd order

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$

$k_1 = F(x_i, y_i)$

$k_2 = F(x_i + p_1 h, y_i + q_{11} k_1 h)$

$a_1 + a_2 = 1 \qquad a_2 p_1 = \frac{1}{2} \qquad a_2 q_{11} = \frac{1}{2}$

Derivation

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = F(x_i, y_i) \quad k_2 = F(x_i + P_1 h, y_i + q_{11} k_1 h)$$

$$y_{i+1} \approx y_i + \underset{\substack{\uparrow \\ y_i' = \frac{dy}{dx}}}{F(x_i, y_i)}h + \frac{F'(x_i, y_i)h^2}{2!} \quad \text{2nd order approximation}$$

$$F'(x_i, y_i) = \frac{\partial F(x,y)}{\partial x} + \frac{\partial F(x,y)}{\partial y}\frac{dy}{dx}$$

$$\boxed{y_{i+1} = y_i + F(x_i, y_i)h + \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}\right)\frac{h^2}{2!}} \quad A$$

now $g(x+r, y+s) = g(x,y) + r\frac{\partial g}{\partial x} + s\frac{\partial g}{\partial y} + \cdots\cdots$

or $k_2 = F(x_i + P_1 h, y_i + q_{11} k_1 h) = \underbrace{F(x_i, y_i) + P_1 h\frac{\partial F}{\partial x} + q_{11} k_1 h \frac{\partial F}{\partial y}}_{k_2} + O(h^2)$

So $y_{i+1} = y_i + a_1 F(x_i, y_i)h + a_2 F(x_i, y_i)h + a_2 P_1 \frac{\partial F}{\partial x} h^2 + a_2 q_{11} F(x_i, y_i) \frac{\partial F}{\partial y} h^2 \ldots$

$$= y_i + (a_1 + a_2) F(x_i, y_i) h + \left[a_2 P_1 \frac{\partial F}{\partial x} + a_2 q_{11} F(x_i, y_i) \frac{\partial F}{\partial y}\right] h^2 + O(h^3)$$

compare to A

$$a_1 + a_2 = 1 \qquad a_2 P_1 = \frac{1}{2} \qquad a_2 q_{11} = \frac{1}{2}$$

if a_2 known

$a_1 = 1 - a_2$

$P_1 = q_{11} = \frac{1}{2a_2}$

$\boxed{a_2 = \frac{1}{2}}$ $\underline{y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h}$ $a_1 = \frac{1}{2}$ $p_1 = q_{11} = 1$

$k_1 = F(x_i, y_i)$

Heun's Method

$k_2 = F(x_i + h, y_i + k_2 h)$

$\boxed{a_2 = 1}$ $a_1 = 0$, $p_1 = q_{11} = \frac{1}{2}$

$\underline{y_{i+1} = y_i + k_2 h}$

$k_1 = F(x_i, y_i)$

$k_2 = F\left(x_i - \frac{h}{2}, y_i + \frac{1}{2}k_1 h\right)$

Ralston's Method $\boxed{a_2 = \frac{2}{3}}$ minimum Truncation error

$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$

$k_1 = F(x_i, y_i)$

$k_2 = F\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$

Lets Compare For $-2x^3 + 12x^2 - 20x + 8.5 = F(x,y) = \frac{dy}{dx}$

Third order

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

$k_1 = F(x_i, y_i)$

$k_2 = F\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$

$k_3 = F(x_i + h, y_i - k_1 h + 2k_2 h)$

4th order

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$k_1 = F(x_i, y_i)$

$k_2 = F(x_i + \frac{1}{2}h, + y_i + \frac{1}{2}k_1 h)$

$k_3 = F(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h)$

$k_4 = F(x_i + h, y_i + k_3 h)$

---

ok we've just been doing more sophisticated integrals, lets get real.

$$\frac{dx}{dT} = -x(T)^3 + \sin(T)$$

def $F(T, x)$
return $-x^3 + \sin(T)$

$a = 0$, $b = 10$, $N = 100$, $h = \frac{b-a}{N}$

T_{points} = arrange (a, b, h)

x_{points} = [ ]

$x = 0.0$

For T in T_{points}
 x_{points}.append (x)
 $k_1 = h \cdot F(T, x)$
 $k_2 = h \cdot F(T + \frac{1}{2}h, x + .5 \cdot k_1)$
 $k_3 = h \cdot F(T + \frac{1}{2}h, x + \frac{1}{2}k_2)$
 $k_4 = h \cdot F(T + h, x + k_3)$
 $x += (k_1 + 2(k_2 + k_3) + k_4)/6$

Coupled Systems ?

$$m\frac{dV}{dT} = -kX \qquad \ddot{X} + \frac{k}{m}X = 0 \qquad X(T) = A\cos(\omega T) + B\sin(\omega T)$$

$$\dot{X}(0) = 1 \qquad X(0) = 1$$

$$\frac{dV}{dT} = -\frac{k}{m}X \qquad \frac{dy_2}{dT} = -\frac{k}{m}y_1$$

$$\dot{X}(T) = -A\omega\sin(\omega T) + B\omega\cos(\omega T)$$

$$V = \frac{dx}{dT} \qquad y_2 = \frac{dy_1}{dT} \qquad A = 1 \qquad B = 1$$

$$\frac{dy_1}{dT} = y_2, \quad \frac{dy_2}{dT} = -\frac{k}{m}y_1 \qquad \underline{F(x) = -wx}$$

$$V_1 = F(x)$$

$$X_1 = V$$

$$V_2 = F\left(x + \frac{h \cdot X_1}{2}\right)$$

$$X_2 = \left(V + \frac{h \cdot V_1}{2}\right)$$

$$V_3 = F\left(x + \frac{h \cdot X_2}{2}\right)$$

$$X_3 = \left(V + \frac{h \cdot V_2}{2}\right)$$

$$V_4 = F(x + h \cdot X_3)$$

$$X_4 = (V + h \cdot V_3)$$

$$V_n = V + (V_1 + 2V_2 + 2V_3 + V_4) \cdot \frac{h}{6}$$

$$X_n = X + (X_1 + 2X_2 + 2X_3 + X_4) \cdot \frac{h}{6}$$

So $\int F(x)\,dx$, $\frac{dx}{dT} = F(x, T)$, $\underbrace{\frac{dV}{dT} = F(x, T) \quad \frac{dx}{dT} = V}_{\text{Second order}}$

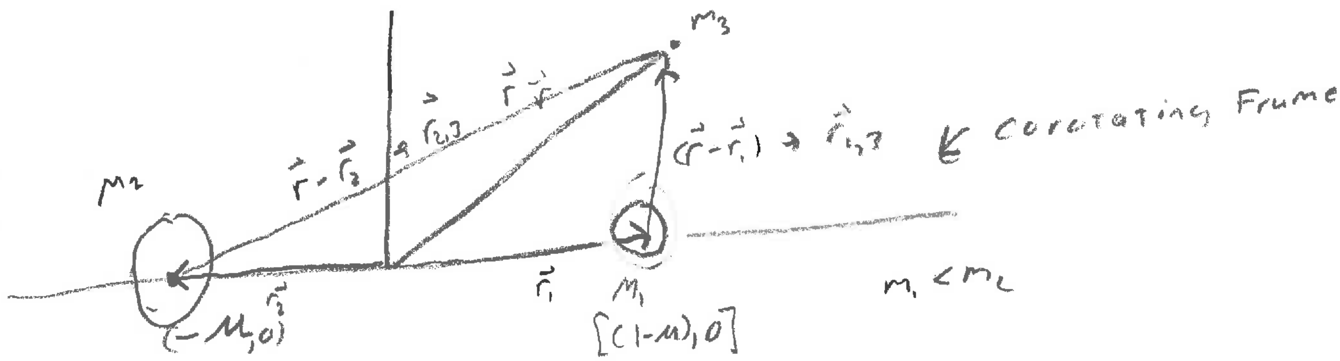
handy?

There's much more

Let's look at error in the SHO case.

$m_2 > m_1 >> m_3$

essentially



Let $M_1 = \mu \quad M_2 = 1-\mu \quad \dfrac{M_1}{M_1+M_2} = M_1$

Show Wikipedia page on Lagrange points

and $|\vec{r}_1| + |\vec{r}_2| = 1 \quad |\vec{r}_2 - \vec{r}_1| = 1$

$\vec{r}_1 = (1-\mu)\hat{x} \quad r_2 = -\mu\hat{x}$

So $\vec{r}-\vec{r}_1 = \vec{r} - (1-\mu)\hat{x} = \vec{r}_{1,3} \quad |\vec{r}_{1,3}|^2 = (x-1+\mu)^2 + y^2$

$\vec{r}-\vec{r}_2 = \vec{r} + \mu\hat{x} = \vec{r}_{2,3} \quad |\vec{r}_{2,3}|^2 = (x+\mu)^2 + y^2$

$$\vec{F}_3 = -\frac{G M_1 m_3}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) - \frac{G M_2 m_3 (\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3}$$

$$\vec{F}_3 = \frac{-G M_1 m_3 \vec{r}_{1,3}}{((x-1+\mu)^2+y^2)^{3/2}} - \frac{G M_2 m_3}{((x+\mu)^2+y^2)^{3/2}}\vec{r}_{2,3}$$

rotating

Need to add centrifugal $a = \dfrac{v^2}{r} = \dfrac{(r\omega)^2}{r} = \omega^2 r = \omega^2 x\hat{x} + \omega^2 y\hat{y}$

and coriolis $= 2\dot{y}\hat{x} - 2\dot{x}\hat{x}$

recalling $\frac{G(M_1 + M_2)}{|r_{12}|^3} = \omega$ keplers law

$$\ddot{x} = 2\dot{y} + x - \frac{\mu}{|r_{13}|^3}(x - 1 + \mu) - \frac{(1-\mu)}{|r_{23}|^3}(x + \mu)$$

$$\ddot{y} = -2\dot{x} + y - \frac{(1-\mu)y}{|r_{23}|^3} - \frac{\mu y}{|r_{13}|^3} = -2\dot{x} + y\left(1 - \frac{\mu}{|r_{13}|^3} - \frac{(1-\mu)}{|r_{23}|^3}\right)$$

huh?

call $\dot{x}_2 = \ddot{x}$ $\quad x_2 = \dot{x}_1$ $\quad \vec{a} = F(x, y, r_{23}, r_{13}, v_x, v_y) = \frac{dv}{d\tau}$

$\dot{y}_2 = \ddot{y}$ $\quad y_2 = \dot{y}_1$ $\quad \vec{v} = \frac{d\vec{r}}{d\tau}$

$\vec{a}$ $\quad V$

ok? $x_0, y_0, V_{x0}, V_{y0} = (\vec{r}_0, \vec{V}_0)$

$$\boxed{\begin{aligned} k\vec{r}1 &= \vec{V}_0 \Delta T \\ k\vec{V}1 &= a(\vec{r}_0, \vec{V}_0, r_{13}, r_{23}) \end{aligned}}$$

$$k\vec{r}2 = \left(\vec{V}_0 + \vec{a}(\vec{r}_0, \vec{V}_0, r_{13}, r_{23})\frac{\Delta T}{2}\right)\Delta T = \left(\vec{V}_0 + \frac{k\vec{V}1}{2}\right)\Delta T$$

$$k\vec{V}2 = a\left(\vec{r}_0 + \frac{\vec{V}_0 \Delta T}{2}, \vec{V}_0 + a(\vec{r}_0, \vec{V}_0, r_{13}, r_{12})\frac{\Delta T}{2}\right)\Delta T = a\left(\vec{r}_0 + \frac{k\vec{r}1}{2}, \vec{V}_0 + \frac{k\vec{V}2}{2}\right)\Delta T$$

ok   ealier

$$\ddot{x} = 2\dot{y} + x - \frac{\mu}{|r_{1,3}|^3}(x - 1 + \mu) - \frac{(1-\mu)}{|r_{2,3}|^3}(x + \mu)$$

$$\ddot{y} = -2\dot{x} + y\left(1 - \frac{\mu}{|r_{1,3}|^3} - \frac{(1-\mu)}{|r_{2,3}|^3}\right)$$

$$\vec{a} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = F(\vec{r}, \vec{V}, |r_{1,3}|, |r_{2,3}|)$$

$$\boxed{\begin{aligned} \vec{kr1} &= \vec{V}_0 \Delta T \\ k\vec{V}1 &= \vec{a}(\vec{r}_0, \vec{V}_0, |r_{1,3}|, |r_{2,3}|)\Delta T \end{aligned}}$$ euler step

$$\vec{kr2} = \left[\vec{V}_0 + a(\vec{r}_0, \vec{V}_0, |r_{1,3}|, |r_{2,3}|)\frac{\Delta T}{2}\right]\Delta T = \left(\vec{V}_0 + \frac{\vec{kV1}}{2}\right)\Delta T$$

$$k\vec{v}2 = \left[\left[a\left(\vec{r}_0 + \frac{\vec{V}_0 \Delta T}{2}\right), \vec{V}_0 + \vec{a}(\vec{r}_0, V_0, r_{13}, r_{23})\frac{\Delta T}{2}\right]\Delta T = \left(a\left(\vec{r}_0 + \frac{\vec{kr2}}{2}, \vec{V}_0 + \frac{\vec{kV1}}{2}, r_{13}, r_{23}\right)\Delta T\right)$$

So $$k\vec{v}2 = a\left(\vec{r}_0 + \frac{\vec{kr2}}{2}, \vec{V}_0 + \frac{\vec{kV1}}{2}, r_{12}, r_{13}\right)\Delta T$$

$$\boxed{\begin{aligned} \vec{kr3} &= \left(\vec{V}_0 + \frac{\vec{kV2}}{2}\right)\Delta T \\ \vec{kV3} &= a\left(\vec{r}_0 + \frac{\vec{kr2}}{2}, \vec{V}_0 + \frac{\vec{kV2}}{2}, r_{13}, r_{23}\right)\Delta T \end{aligned}}$$

$$\boxed{\begin{aligned} \vec{kr4} &= \left(\vec{V}_0 + \vec{kV3}\right)\Delta T \\ k\vec{V}4 &= a\left(\vec{r}_0 + \vec{kr3}, \vec{V}_0 + \vec{kV3}, r_{13}, r_{23}\right)\Delta T \end{aligned}}$$

$$\vec{r}_n = \vec{r}_0 + \frac{1}{6}\vec{kr1} + \frac{1}{3}\vec{kr_2} + \frac{1}{3}k\vec{r3} + \frac{1}{6}\vec{kr4}$$

$$\vec{V}_n = \vec{V}_0 + \frac{1}{6}k\vec{V1} + \frac{1}{3}k\vec{V2} + \frac{1}{3}k\vec{V3} + \frac{1}{6}kV4$$

Switched a bit by putting Δt into k's

Also, 1 swindle. What was it?

$r_{1,3}$, $r_{2,3}$ really need to be updated

	k1	$r_{13}(x, y)$	$r_{23}(\ )$
So at	k2	$r_{13}\left(x + \frac{kx_1}{2}, y + \frac{ky_1}{2}\right)$	$r_{23}\ldots$
	k3	$r_{13}\left(x + \frac{kx_2}{2}, y + \frac{ky_2}{2}\right)$	$r_{23}\ldots$
	k4	$r_{13}(x + kx3, y + ky3)$	$r_{23}\ldots$

Outline

define

```
rad13(x, y, u)
    return sqrt((x-1+u)^2+y^2)

rad23(x, y, u)
    return sqrt((x+u)^2 + y^2))
```

xddot
ydddot   see example code

$$\vec{X}_n = \vec{X}_0 + \frac{1}{6} kX1 + \frac{1}{3} kX_2 + \frac{1}{3} kX3 + \frac{1}{6} k\vec{X}4$$

$$\vec{V}_n = \vec{V}_0 + \frac{1}{6} kv1 + \frac{1}{3} k\,v2 + \frac{1}{3} kv3 + \frac{1}{6} kv4$$

$|r_1| + |r_2| = 1, \quad |r_2 - r_1| = 1, \quad m_1 + m_2 = 1,$

$\vec{r}_1 = 1-\mu$	$m_1 = \mu$
$\vec{r}_2 = -\mu$	$m_2 = 1-\mu$

$$\mu = \frac{m_1}{m_1+m_2} = m_1$$

Ex $\mu = .2$ $\quad m_1 = .2 \quad r_1 = .8$

$\qquad\qquad m_2 = .8 \quad r_2 = -.2$

$$r_1^2 = (x - 1 + \mu)^2 + y^2$$

$$r_2^2 = (x + \mu)^2 + y^2$$

Now $G = 1$, $w = 1$ $\quad \ddot{\vec{r}} - w\vec{r} - 2w\vec{v} = \frac{-\mu}{r_1^3}(\vec{r} - \vec{r}_1) - \frac{(1-\mu)}{r_2^3}(\vec{r} - \vec{r}_2)$

$$\ddot{X} - \vec{r} - 2\vec{V} = \frac{-\mu}{r_1^3}$$

Example

$\vec{a} = F(\vec{x}, \vec{v})$

$\vec{v} = \vec{a}\, dT$ (1st order)

$\vec{x} = \vec{v}\, dT$ (1st order) → Euler Not Energy Conserving

$a_x = F(v_y, x, \mu, r_1, r_2)$

$a_y = F(v_x, y, \mu, r_1, r_2)$

Better

$x_0, y_0, v_{x0}, v_{y0} = (\vec{x}_0, \vec{v}_0)$

$k\vec{x1} = \vec{v}_0 \Delta T$

$k\vec{v1} = a(\vec{x}_0, \vec{v}_0)\Delta T$

$k\vec{x2} = \left(\vec{v}_0 + \frac{a(\vec{x}_0, \vec{v}_0)\Delta T}{2}\right)\Delta T = \left(\vec{v}_0 + \frac{k\vec{v1}}{2}\right)\Delta T$

$k\vec{v2} = \left(a\left(\vec{x}_0 + \frac{\vec{v}_0 \Delta T}{2}, \vec{v}_0 + \frac{a(\vec{x}_0, \vec{v}_0)\Delta T}{2}\right)\right)\Delta T = a\left(\vec{x}_0 + \frac{k\vec{x1}}{2}, \vec{v}_0 + \frac{k\vec{v1}}{2}\right)\Delta T$

$k\vec{x3} = \left(\vec{v}_0 + \frac{k\vec{v2}}{2}\right)\Delta T$

$k\vec{v3} = a\left(\vec{x}_0 + \frac{k\vec{x2}}{2}, \vec{v}_0 + \frac{k\vec{v2}}{2}\right)\Delta T$

$k\vec{x4} = \left(\vec{v}_0 + k\vec{v3}\right)\Delta T$

$k\vec{v4} = a\left(\vec{x}_0 + k\vec{x3}, \vec{v}_0 + k\vec{v3}\right)\Delta T$

$\vec{x}$