Differential Equations

Easy Start 2nd order $\vec{F} = m\vec{a}$ using Forward Euler

$$\frac{\vec{F}}{m} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} \qquad \text{Let } \vec{F} = 0\hat{i} + (-g)\hat{j} \qquad g = 9.8\,\frac{m}{s^2}$$

Simple
$$\vec{r}(T) = (X_0 + V_{x0}T)\hat{i} + (y_0 + V_{y0}T - \tfrac{1}{2}gT^2)\hat{j}$$
$$\vec{V}(T) = V_{x0}\,\hat{i} + (V_{y0} - gT)\,\hat{j}$$

Now numerically

$$\frac{d\vec{v}}{dT} = 0\hat{i} - \frac{g}{m}\hat{j} \qquad \text{remember } \frac{F(x+h) - F(x)}{h} = \frac{dF}{dx}$$

How about
$$\lim_{\Delta T \to 0} \frac{V(T+\Delta T) - V(T)}{\Delta T} = \frac{dV}{dT}$$

or
$$\frac{\vec{V}(T+\Delta T) - \vec{V}(T)}{\Delta T} = 0\hat{i} - \frac{g}{m}\hat{j} \qquad \text{2 equations}$$

The 1st is trivial $V_x(T) = V_{x0}$ $\quad X(T) = X_0 + V_{x0}T$

The second? $V_y(T+\Delta T) - V_y(T) = -\frac{g}{m}\Delta T$ $\quad$ Skip Subscript

$$\underbrace{V(T_0 + \Delta T)}_{V(T_1)} = V(T_0) \underbrace{- \frac{g}{m}\Delta T}_{\Delta V} \qquad V(T_1) = V(T_0) + \Delta V$$

$$V(T_2) = V(T_1) - \frac{g}{m}\Delta T \,, \quad V(T_3) = V(T_2) - \frac{g\Delta T}{m}$$

got it? Code it! use np.append

how about $X(T)$?

$$\frac{dx}{dT} = V(T)$$

$$\frac{\Delta y}{\Delta T} = V(T) \qquad y(T_0 + \Delta T) = \overbrace{Y(T_0) + V(T_0)\Delta T}^{X(T_1)}$$

$$\text{or} \quad Y(T_0) + \left[\frac{V(T_0) + V(T_1)}{2}\right]\Delta T$$

$$Y(T_1 + \Delta T) = Y(T_1) + \nearrow \quad \text{better} \quad \nwarrow \overline{V(T)}$$

$T_0 \to T_1$, $T_1 \to T_2$

How To pick ΔT?

Harder

$$\frac{d\vec{V}}{dT} = \frac{1}{2}\frac{\rho}{m} v^2 C_d A \hat{v} - \frac{g}{m}\hat{j} = -bv^2\hat{v} - \frac{g}{m}\hat{j} \qquad b = \frac{1}{2m}\rho C_d A$$

$\hat{i}, \hat{j} \uparrow$

$$-bv^2\hat{v}? \qquad -b\,(V_x^2 + V_y^2)\frac{(V_x\hat{i} + V_y\hat{j})}{\sqrt{V_x^2 + V_y^2}}$$

$$\frac{d\vec{V}}{dT} = -b\left(\sqrt{V_x^2 + V_y^2}\right)V_x\hat{i} - \left(b\,(V_x^2 + V_y^2)^{1/2} V_y + \frac{g}{m}\right)\hat{j}$$

$$V_x(T + \Delta T) = V_x(T) - b\,(V_x^2 + V_y^2)^{1/2} V_x \Delta T$$

$$V_y(T + \Delta T) = V_y(T) - \left[b(V_x^2 + V_y^2)V_y + \frac{g}{m}\right]\Delta T$$

$$y(T + \Delta T) = y(T) + \overline{V_y(T)}\Delta T$$

$$X(T + \Delta T) = X(T) + \overline{V_x(T)}\Delta T$$

need X_0, Y_0, V_{x0}, V_{y0}

Harder

$$\frac{\partial\, \text{Temp}}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T = T(x, t)$$

$$\vec{F} = m\frac{d\vec{v}}{dT} \qquad \vec{F} = \underbrace{-\frac{1}{2}\rho C_d A}_{b \text{ if } \rho = \text{constant}} V^2\hat{v} - g\hat{j}$$

$$V^2\hat{v} = \frac{(V_x^2 + V_y^2)(V_x\hat{i} + V_y\hat{j})}{\sqrt{V_x^2+V_y^2}} = \sqrt{V_x^2+V_y^2}\,(V_x\hat{i} + V_y\hat{j})$$

$X_0 = 0 \qquad y_0 = 4346\,m$

Stop at $y = 0$

$V_{x0} = 250\frac{m}{s} \qquad V_{y0} = 0\frac{m}{s}$

Speed at $y=0$ no drag, drag, exponentially stratified atmosphere

X vs y

$$\frac{d\vec{V}}{dT} = \left(\frac{dV_x}{dT}\hat{i} + \frac{dV_y}{dT}\hat{j}\right)\frac{1}{m}$$

$$\frac{dV_x}{dT}\hat{i} = \left(-\frac{1}{2}\frac{\rho C_d A}{m}\sqrt{V_x^2+V_y^2}\right)V_x\hat{i}$$

$$\frac{dV_y}{dT}\hat{j} = -\left[\left(\frac{1}{2}\frac{\rho C_d A}{m}\sqrt{V_x^2+V_y^2}\right)V_y + g\right]\hat{j}$$

$$\rho(y) = \rho_0 e^{-\frac{kT}{Mg}y} \qquad H^{-1}$$

$k = 1.38 \cdot 10^{-23}\,J \cdot K^{-1}$

$T = 8250\,m$

$g = 9.8$

$$\rho(y) = \rho_0 e^{-\frac{y}{H}}$$

M = mean molecular mas

$H \approx 8250\,m \qquad \rho_0 = 1.225\,\frac{kg}{m^3}$

$V(T_i + \Delta T) = V(T_i) + \frac{F}{m}(\vec{V}(T_i), y(T_i))\,\Delta T$

Time marching not Space

V_{x0} [ ]

↓

V_{x1} [ ]

↓

" "

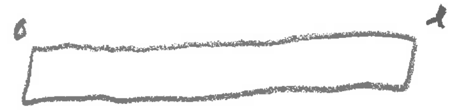
$\Delta T = ?$ $\quad y_F = \frac{1}{2} g T^2$ $\quad T = .1 \quad \Delta x = 5\text{cm}$ good enough

$\Delta y_{total} = 270\text{ m} \sim 5500$ Steps For No drag how many does it take

$\frac{dx}{dt} = V \qquad X(T_i + \Delta T) = X_i + V_i \Delta T$

Check $\quad mg = \frac{1}{2} \rho C_d A V^2 \qquad V_T = \left(\frac{2mg}{\rho C_D A}\right)^{1/2}$ $\quad m = 75$, $C = .5$, $A = .1$ $\quad V \sim 155 \frac{m}{s}$

Let $T = \Sigma C_n e^{i(kx - \omega t)}$ 1D



$\frac{\partial T}{\partial t} = -i\omega T \quad \alpha \frac{\partial^2 T}{\partial x^2} = -\alpha k^2 T$ So $i\omega = \alpha k^2$

Now $\frac{\omega}{k} = -i\alpha k$ dispersion relation $\frac{\omega}{k}$ = Phase Velocity

$\frac{d\omega}{dk} = -2i\alpha k \neq \frac{\omega}{k}$ = group Velocity dispersive

Sound $\omega = k C_s^2$ ← Sound Speed $\frac{d\omega}{dk} = \frac{\omega}{k}$ non-dispersive

How To Solve?

$T(x,t) = \Sigma C_n e^{i(kx + i\alpha k^2 t)}$ $k = \frac{2\pi}{\lambda}$ $\omega = 2\pi\nu$

$T(x,t) = \Sigma C_n e^{ikx} e^{-\alpha k^2 t}$ decays with time

Shorter waves = faster decay

Solution? IC'S, BC'S 1 IC 2 BC

$\frac{\partial T}{\partial t}$ $\frac{\partial^2 T}{\partial x^2}$

1 initial Shape $T(x,0) = g(x)$ Starting Shape

2 $T(0,t) = 0 \rightarrow \Sigma [a_n \cos(k \cdot 0) + b_n \sin(k \cdot 0)] e^{-\alpha k^2 t} = 0$

$a_n = 0$

3. $T(l,t) = 0 \rightarrow \Sigma b_n \sin(k \cdot l) e^{-\alpha k^2 t} = 0$

So $\sin(kl) = 0$ or $kl = n\pi$

Now $T(x,t) = \Sigma b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha k^2 t}$

just need b_n get from $g(x)$ and Fourier's "Trick"

Properly Normalized $A\,\sin\left(\frac{n\pi x}{\ell}\right)$ Form a complete orthonormal set,

$$F(x) = \sum b_n A \sin\left(\frac{n\pi x}{\ell}\right) \;,\quad \int_0^{\ell} A^2 \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)dx = \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$$ Think $\hat{i}, \hat{j}, \hat{k}$ Example

$A = ?$ $$\int_0^{\ell} A^2 \sin^2\left(\frac{n\pi x}{\ell}\right) dx = 1$$

$$A^2 \int_0^{\ell} \sin^2(kx)\,dx = 1$$

let $u = \frac{n\pi x}{\ell} = kx$

$du = k\,dx$

$x=0 \quad u=0 \qquad x=\ell \quad u=n\pi$

$$\frac{A^2}{k}\int_0^{n\pi} \sin^2(u)\,du = 1 \qquad \frac{A^2}{k}\int_0^{n\pi}\left(\frac{1}{2} - \cancel{\frac{\cos(2u)}{2}}\right)du = 1$$

0

$$\frac{A^2}{k}\,\frac{u}{2}\Big|_0^{n\pi} = \frac{n\pi}{2}\cdot\frac{A^2\ell}{n\pi} = 1 \qquad A = \sqrt{\frac{2}{\ell}}$$

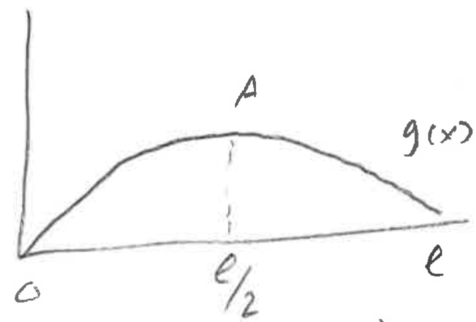
$b_n = ?$ $$g(x) \cong \sum b_n \sin\left(\frac{n\pi x}{\ell}\right)$$

$$\int g(x)\sin\left(\frac{m\pi x}{\ell}\right)dx = \int \sum b_n \underbrace{\sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)}_{\frac{\ell\cdot\delta_{mn}}{2}}dx$$

$$\frac{\ell b_n}{2} = \int_0^{\ell} g(x)\, \mathrm{SIN}\left(\frac{m\pi x}{\ell}\right) dx$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} g(x)\, \mathrm{SIN}\left(\frac{m\pi x}{\ell}\right) dx$$

Concrete Example



$$ax^2 + bx + c = g(x)$$

$$a(0)^2 + b(0) + c = 0$$

$$c = 0 \quad \underline{1}$$

$$a\ell^2 + b\ell = 0$$

$$a\ell + b = 0 \qquad A = -\frac{b}{\ell} \quad \underline{2}$$

$$-\frac{b}{\ell}\left(\frac{\ell^2}{4}\right) + b\frac{\ell}{2} = A$$

$$-\frac{b\ell}{4} + \frac{b\ell}{2} = A$$

$$\frac{b\ell}{4} = A \qquad b = \frac{4A}{\ell} \quad \underline{3}$$

$$g(x) = -\frac{4A}{\ell^2}x^2 + \frac{4A}{\ell}x$$

Let $A = 1$ $\ell = 1$ $g(x) = -4x^2 + 4x$

$$b_n = 2\int_0^1 (-4x^2 + 4x)\, \mathrm{SIN}\left(\frac{m\pi x}{1}\right) dx$$

$$b_n = 2\left[\underbrace{\frac{8}{m^3\pi^3} - \frac{8\cos(m\pi)}{m^3\pi^3}}_{\substack{0 \text{ if } m = 0,2,4,\dots \\ 16 \text{ if } m = 1,3,5\dots}} - \frac{4m\pi\, \mathrm{SIN}(m\pi)}{m^3\pi^3}\right] \quad m = 0,1,2,3\dots$$

↑ always zero

$$T(x,t) = 2 \cdot \sum_{m \text{ odd}} \frac{16}{m^3 \pi^3} \sin\left(\frac{m \pi x}{\ell}\right) e^{-\alpha k^2 t}$$

huh? α? $\alpha k^2 t$ is dimensionless

So α has units $\frac{s}{m^2}$ or diffusion rate per meter

when $t = \frac{1}{\alpha k^2}$ $e^{-\alpha k^2 t} = e^{-1}$ $T_0 = \frac{1}{\alpha k^2}$ e folding time

Maybe computer time

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \frac{T_j^{n+1} - T_j^n}{\Delta t} = \alpha \left[\frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{(\Delta x)^2} \right]$$

time (n), space (j); 1st order FD; 2nd order CD

$$T_j^{n+1} = T_j(t+\Delta t) \qquad T_j^{n+1} = T_j^n + \frac{\alpha \Delta t}{(\Delta x)^2} \left[T_{j+1}^n - 2T_j^n + T_{j-1}^n \right] + O(\Delta t) + O(\;)$$

$\Delta t = ?$ Look at amplification factor $\frac{\epsilon_j^{n+1}}{\epsilon_j^n} = G$

A = analytic solution, D = exact solution of difference error

N = Numerical solution from a computer with finite accuracy

Discretization error = A − D

Round off error = ϵ = N − D

N = D + ϵ Exact + Round off error

We are Solving N Let $u \to N = D + \epsilon$ and $j \to i$

$$\frac{D_i^{n+1} + \epsilon_i^{n+1} - D_i^n - \epsilon_i^n}{k\Delta T} = \frac{D_{i+1}^n + \epsilon_{i+1}^n - 2D_i^n - 2\epsilon_i^n + D_{i-1}^n + \epsilon_{i-1}^n}{(\Delta x)^2}$$

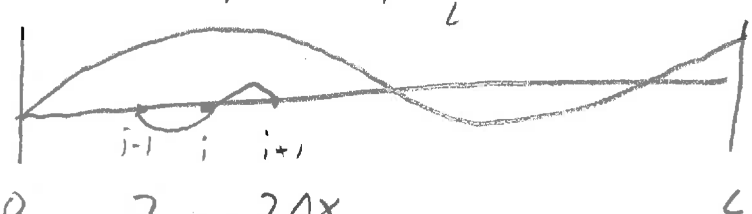
by definition $\dfrac{D_i^{n+1} - D_i^n}{k\Delta T} = \dfrac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{(\Delta x)^2}$

We are left with $\dfrac{\epsilon_i^{n+1} - \epsilon_i^n}{k\Delta T} = \dfrac{\epsilon_{i+1}^n - 2\epsilon_i^n + \epsilon_{i-1}^n}{(\Delta x)^2}$

Want $\left|\dfrac{\epsilon_i^{n+1}}{\epsilon_i}\right| = |G| \leq 1$ or error GROWS!!

$\epsilon(x) = \sum_m A_m e^{ik_m x}$ $\quad k_m = \left(\frac{2\pi}{L}\right)m \quad m = 1, 2, 3$

$\lambda_{max} = L \quad k_1 = \frac{2\pi}{L} \cdot 1$ $\quad m \uparrow \; \lambda \downarrow$



$\lambda_{min} = 2\Delta x$

↖ minimum resolvable mode

if $N+1$ gridpoints N intervals $\Delta x = \frac{L}{N}$

$\lambda_{min} = \frac{2L}{N}$ For a grid with $N+1$ pts

$k_{min} = \dfrac{2\pi}{\lambda_{min}} = \dfrac{2\pi}{2L/N} = \underline{\dfrac{2\pi}{L}\dfrac{N}{2}}$ So $\epsilon(x) = \sum_{m=1}^{N/2} A_m e^{ik_m x}$

$$\epsilon(x,T) = \sum_{m=1}^{N/2} A_m(T) e^{ik_m x} = \sum_{m=1}^{N/2} e^{aT} e^{ik_m x}$$

Lets assume $\epsilon(x,T) = e^{aT} e^{ikx}$ just one wave

1: Then

$$\frac{\overset{\epsilon_i^{n+1}}{e^{a(T+\Delta T)} e^{ikx}} - \overset{\epsilon_i^n}{e^{aT} e^{ikx}}}{k\Delta T} = \frac{\overset{\epsilon_{i+1}^n}{e^{aT} e^{ik(x+\Delta x)}} - \overset{\epsilon_i^n}{2e^{aT} e^{ikx}} + \overset{\epsilon_{i-1}^n}{e^{aT} e^{ik(x-\Delta x)}}}{(\Delta x)^2}$$

2: divide by $e^{aT} e^{ikx}$

3: $$\frac{e^{a\Delta T} - 1}{k\Delta T} = \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{(\Delta x)^2}$$

4: $$e^{a\Delta T} = 1 + \frac{k\Delta T}{(\Delta x)^2}\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$ " $\frac{e^{ikx} + e^{-ikx}}{2} = \cos(x)$ "

5: $$e^{a\Delta T} = 1 + \frac{2k\Delta T}{(\Delta x^2)}\left(\cos(k\Delta x) - 1\right)$$ " $\sin^2\left(\frac{kx}{2}\right) = \frac{1-\cos(kx)}{2}$

6: $$e^{a\Delta T} = 1 - \frac{4k\Delta T}{(\Delta x)^2} \sin^2\left(\frac{k\Delta x}{2}\right)$$

7: $$\frac{\epsilon_i^{n+1}}{\epsilon_i} = \frac{e^{a(T+\Delta T)} e^{ikx}}{e^{aT} e^{ikx}} = e^{a\Delta T} = G$$

8: Need $$\left|\frac{\epsilon_i^{n+1}}{\epsilon_i}\right| = \left|e^{a\Delta T}\right| = \left|1 - \frac{4k\Delta T}{(\Delta x)^2} \sin^2\left(\frac{k\Delta x}{2}\right)\right| \leq 1$$

or $$1-\frac{4k\Delta T}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)\le 1$$

always positive always True

and

$$1-\frac{4k\Delta T}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)\ge -1$$ multiply by (-1)

So $$\frac{4k\Delta T}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)-1\le 1$$

From which $$\frac{4k\Delta T}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)\le 2$$

or $$\frac{k\Delta T}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)\le \frac{1}{2}$$

max=1

So $$\frac{k\Delta T}{(\Delta x)^2}\le\frac{1}{2}$$

or $$\Delta T\le\frac{(\Delta x)^2}{2k}$$

Δx, k Set ΔT $\Delta x\downarrow$ $\Delta T\downarrow$ $k\uparrow$ $\Delta T\downarrow$

generically, if a equation deals with

wave propagation $\Delta T \leq C_{Fl} \dfrac{\Delta x}{V_{max}}$ $C_{Fe} \sim .5 \rightarrow .8$