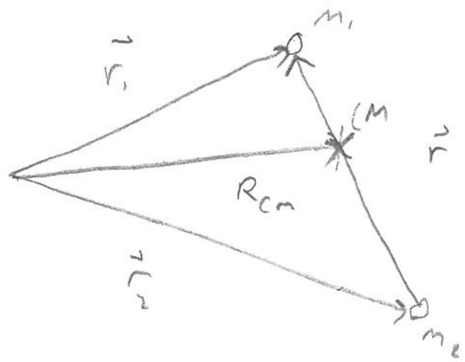
Chapter 8

Central Force Motion

Reduced Mass → 2 body to 1 body



$\vec{r}_2 + \vec{r} = \vec{r}_1 \qquad \vec{r}_2 = \vec{r}_1 - \vec{r}$

$\vec{r} = \vec{r}_1 - \vec{r}_2$

$|\vec{r}| = |\vec{r}_1 - \vec{r}_2|$

$L = \frac{1}{2} m_1 |\dot{r}_1|^2 + \frac{1}{2} m_2 |\dot{r}_2|^2 - U(r)$

Let $\vec{R}_{cm} = 0$

Then $\dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = 0$ or $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$

Essentially we shift our origin

Then $m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 = m_1(\vec{r}_2 + \vec{r}) + m_2(\vec{r}_2) = 0$

or $(m_1 + m_2)(\vec{r}_2) = -m_1 \vec{r}$ $\quad \boxed{\vec{r}_2 = -\dfrac{m_1}{(m_1 + m_2)} \vec{r}}$

and $m_1 \vec{r}_1 + m_2(\vec{r}_1 - \vec{r}) = 0 \quad (m_1 + m_2)\vec{r}_1 = m_2 \vec{r}$

$$\boxed{\vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}}$$

If $m_2 = m_1$

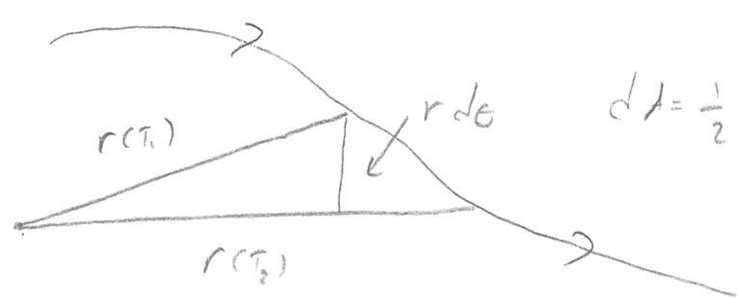
$$\vec{r}_1 = \frac{\frac{m_2}{m_1}}{\frac{m_1}{m_1} + \frac{m_2}{m_1}} = \frac{1}{2}\vec{r} \qquad \vec{r}_2 = -\frac{1}{2}\vec{r}$$

orbit

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\dot{P}_\theta = \frac{\partial L}{\partial \theta} = 0 \to \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$ angular momentum is conserved

$$P_\theta = \text{constant} = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = |L|$$



$$dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$\vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r}$ and $\vec{r}_2 = \frac{-m_1}{m_1 + m_2} \vec{r}$

gives $L = \frac{1}{2} \mu \left[|\dot{r}|^2 \right] - U(r)$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Solve $r(t)$ get $\vec{r}_1(t)$ and $\vec{r}_2(t)$ for the price of one

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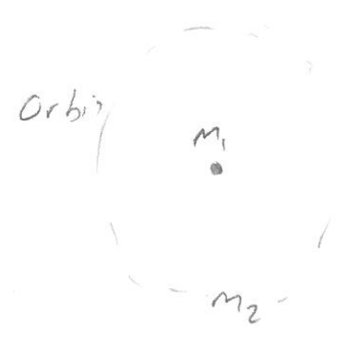
what if $m_1 >> m_2$

$$\mu = \frac{\frac{m_1 m_2}{m_1}}{\frac{m_1}{m_1} + \frac{m_2}{m_1}} = \frac{m_2}{1 + \text{Tiny}} = m_2$$

$$\vec{r}_1 = \frac{\frac{m_2}{m_1}}{\frac{m_1}{m_1} + \frac{m_2}{m_1}} \vec{r} = 0 \qquad \vec{r}_2 = - \frac{\frac{m_1}{m_1}}{\frac{m_1}{m_1} + \frac{m_2}{m_1}} \vec{r} = -\vec{r}$$

So $|\vec{r}|$ essentially $|\vec{r}_2|$



$\frac{1}{2} r^2 \dot{\theta} = \frac{|L|}{2\mu} =$ constant

Keplers Second Law

$$\frac{dA}{dt} = \frac{|L|}{2\mu} =$$ Equal Areas in Equal Times

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Now $E = T + U$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{|L|^2}{\mu r^2} + U(r)$$

Motion?

Motion

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)$$

$$2(E - U(r)) = \mu \dot{r}^2 + \frac{\ell^2}{\mu r^2}$$

$$\pm\left(\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}\right)^{1/2} = \frac{dr}{dt} = \dot{r}$$

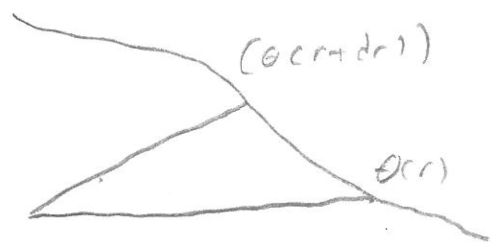
Trick

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

but $\dot{\theta} = \frac{\ell}{\mu r^2}$

So $\frac{\dot{\theta}}{\dot{r}}dr = \frac{\ell}{\mu r^2}\cdot\frac{dr}{\dot{r}} = d\theta$

$$\pm\frac{\ell}{\mu r^2}\cdot\left(\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}\right)^{-1/2} dr = \theta(r)$$



$$\pm\frac{\ell}{\mu r^2}\frac{1}{\sqrt{\frac{2(E-U)}{\mu} - \frac{\ell^2}{\mu^2 r^2}}}dr = \theta(r)$$

$$\pm\frac{\ell}{r^2}\cdot\frac{1}{\sqrt{2\mu(E-U) - \frac{\ell^2}{r^2}}}dr = \theta(r)$$

$$\pm\int\frac{\ell/r^2\, dr}{\sqrt{2\mu\left[E - U - \frac{\ell^2}{2\mu r^2}\right]}} = \theta(r)$$

$$\theta(r) = \int \frac{\pm(\ell/r^2)\,dr}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}$$

$\dot{\theta} \propto L \leftarrow$ constant same

So $\theta \nearrow$ or $\downarrow$ only

Only analytic for $F(r) \propto r^n$ when $n = 1, -2, -3$ $\leftarrow$ HW?

Can also do via Lagrangian $\rightarrow$

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \mu\ddot{r} \qquad \frac{\partial L}{\partial r} = \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r}$$

$$\mu\ddot{r} - \mu r\dot{\theta}^2 = -\frac{\partial U}{\partial r} = F(r)$$

Let $u = \frac{1}{r}$ $\qquad \dot{\theta} = \frac{\ell}{\mu r^2}$

a.) $$\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = \underline{\underline{-\frac{\mu}{\ell}\dot{r}}}$$

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b.) $$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left[\frac{du}{d\theta}\right] = \frac{d}{d\theta}\left(-\frac{\mu}{\ell}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{\ell}\dot{r}\right) = -\frac{1}{\dot{\theta}}\frac{\mu}{\ell}\ddot{r}$$

$$\underline{\underline{\frac{d^2u}{d\theta^2} = -\frac{\mu^2 r^2}{\ell^2}\ddot{r}}}$$

Then

$$\ddot{r} = -\frac{\ell^2}{\mu^2}\overset{\frac{1}{r^2}}{u^2}\frac{d^2u}{d\theta^2} \quad , \quad r\dot{\theta}^2 = \frac{\ell^2 u^3}{\mu^2}$$

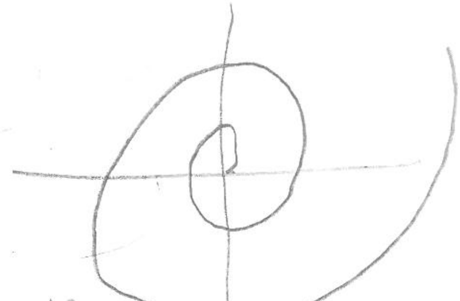
Then $\mu\left[-\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\theta^2} - \frac{\ell^2 u^3}{\mu^2}\right] = F\left(\frac{1}{u}\right)$

or $\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F(1/u)$

or $\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu}{\ell^2}\, r^2 F(r) \rightarrow$ want $r(\theta)$

force which gives rise to orbit

Ex What Force allows $r = ke^{\alpha\theta}$



$\frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$

$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{-\alpha}{k}e^{-\alpha\theta} \qquad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k}e^{-\alpha\theta} = \frac{\alpha^2}{r}$

$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{r}(\alpha+1) = -\frac{\mu}{\ell^2}\, r^2 F(r)$

$F(r) = \underset{n=3}{\frac{-1}{r^3}}\,\frac{\ell^2}{\mu}(\alpha+1)$

$r(T)$ and $\theta(T)$

$$\dot{\theta} = \frac{l}{\mu r^2} = \frac{l}{\mu k^2 e^{2\alpha\theta}} = \frac{d\theta}{dT} \rightarrow \frac{l}{\mu k^2} dT = e^{2\alpha\theta} d\theta$$

$u = 2\alpha\theta \quad du = 2\alpha\, d\theta$

$$\frac{lT}{\mu k^2} = \frac{1}{2\alpha} e^{2\alpha\theta} + C$$

$$\rightarrow \frac{2\alpha l T}{\mu k^2} + \underbrace{C'}_{-2\alpha C} = e^{2\alpha\theta} \rightarrow \frac{1}{2\alpha} \ln\left[\frac{2\alpha l T}{\mu k^2} + C\right] = \theta(T)$$

Then $r = ke^{\alpha\theta} \rightarrow r^2 = k^2 e^{2\alpha\theta} \rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$

So $\frac{r^2}{k^2} = \frac{2\alpha l T}{\mu k^2} + C' \rightarrow r^2 = \frac{2\alpha l T}{\mu} + D$

$$r(T) = \pm \sqrt{\frac{2\alpha l}{\mu} T + D} \qquad n = -3$$

Energy? $E = T + U$

U? $U(r) = -\int \vec{F} \cdot d\vec{r} = +\frac{l^2}{\mu}(\alpha^2+1) \int r^{-3} dr = \frac{-l^2}{2\mu} \frac{(\alpha^2+1)}{r^2}$

$\dot{r}$? $\quad \dot{\theta} = \frac{d\theta}{dr}\frac{dr}{dt} = \frac{\ell}{\mu r^2} \rightarrow \quad \dot{r} = \frac{dr}{d\theta}\frac{\ell}{\mu r^2}$

$$\dot{r} = \alpha k e^{\alpha\theta}\frac{\ell}{\mu r^2} = \frac{\alpha \ell}{\mu r}$$

And $\frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{\ell^2(\alpha^2+1)}{2\mu r^2} = E$

$$\frac{\alpha^2\ell^2}{\mu^2 r^2} \rightarrow \frac{1}{2}\frac{\alpha^2\ell^2}{\mu r^2} + \frac{\ell^2}{2\mu r^2} - \frac{1}{2}\frac{\alpha^2\ell^2}{\mu r^2} - \frac{\ell^2}{2\mu r^2} = E = 0$$

1 2 3 4

Central Field orbits

$$\dot{r} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}} = 0 \text{ when } = 0$$

Turning points between r_{min} and r_{max}

If only one root orbit is circular

If closed orbit repeats path if not, not.

See Figure 8-4

$\Delta\theta$ in $r_{min} \to r_{max} \to r_{min} = 2 \displaystyle\int_{r_{min}}^{r_{max}} \frac{l/r^2 \, dr}{\sqrt{2\mu\left[E - U - \frac{l^2}{2\mu r^2}\right]}}$

Closed for $\Delta\theta = 2\pi \left(\frac{a}{b}\right)$]- integers

Closed noncircular $n = -2, +1$ ← harmonic oscillator

$U(r) \propto r^{n+1}$ ↑ inverse square law gravity

r^{-1}, r^2

Centrifugal Energy, Potential with effective potential

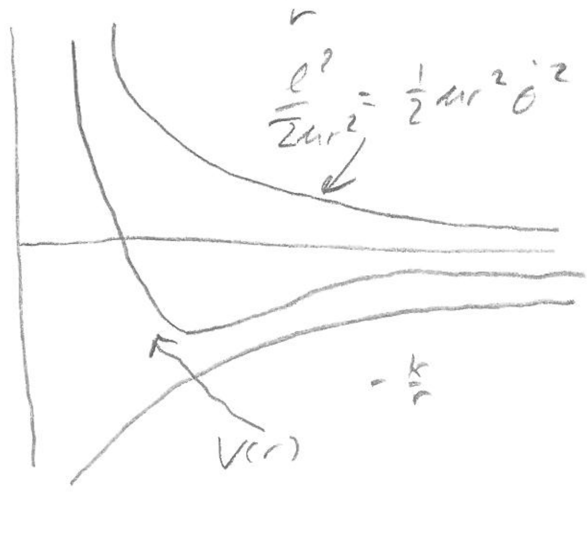
$\frac{1}{2}\mu r^2\dot{\theta}^2 = \frac{l^2}{2\mu r^2} \rightarrow U_c = \frac{l^2}{2\mu r^2}$ $\quad F_c = -\frac{\partial U}{\partial r} = \frac{l^2}{\mu r^3} = \mu r\dot{\theta}^2$

↑ Centripetal Force

$V(r) = U(r) + \frac{l^2}{2\mu r^2}$

↑ centrifugal barrier when $F(r) = -\frac{k}{r^2}$

and $U(r) = -\frac{k}{r}$ so $V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$



$E_1 \to$ Free
Scattering
E ↓ bound
ellipse
E_2
E_3
r_2
3
r_4
$\epsilon=0$
orbit circular

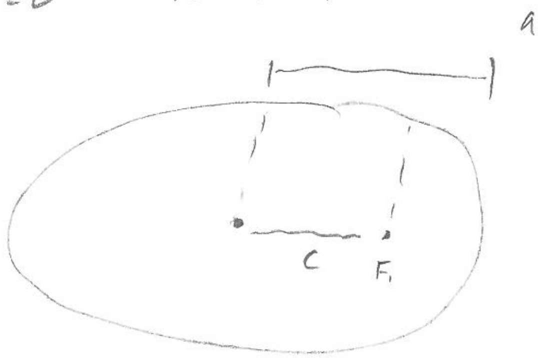
$$c = \frac{\alpha \epsilon}{1-\epsilon^2} \rightarrow \quad b = \left(\frac{\alpha^2}{(1-\epsilon^2)^2} - \frac{\alpha^2 \epsilon^2}{1-\epsilon^2} \right)^{1/2}$$

$$b = \left(\frac{1}{(1-\epsilon^2)^2} \cdot \alpha^2 (1-\epsilon^2) \right)^{1/2} = \frac{\alpha}{\sqrt{1-\epsilon^2}}$$

$$r_{min} = \frac{\alpha}{1+\epsilon} \quad \text{Perihelion} \qquad r_{max} = \frac{\alpha}{1-\epsilon} \quad \text{aphelion}$$

$$= a(1-\epsilon) \qquad\qquad = a(1+\epsilon)$$

if $\epsilon = 0$ $\quad r = a \rightarrow$ circle



$$c = \frac{\alpha \epsilon}{1-\epsilon^2} = \epsilon a$$

or $\frac{c}{a} = \epsilon$

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recall $\frac{dA}{dt} = \frac{\ell}{2\mu} \rightarrow \quad dt = \frac{2\mu}{\ell} dA$

$$\int_0^{\tau} dt = \frac{2\mu}{\ell} \int_0^A dA \rightarrow \tau = \frac{2\mu}{\ell} A = \frac{2\mu}{\ell} \pi a b$$

$$\tau = \frac{2\mu}{\ell} \pi a b = \pi k \sqrt{\frac{\mu}{2}} \, |E|^{-3/2}$$

Orbits

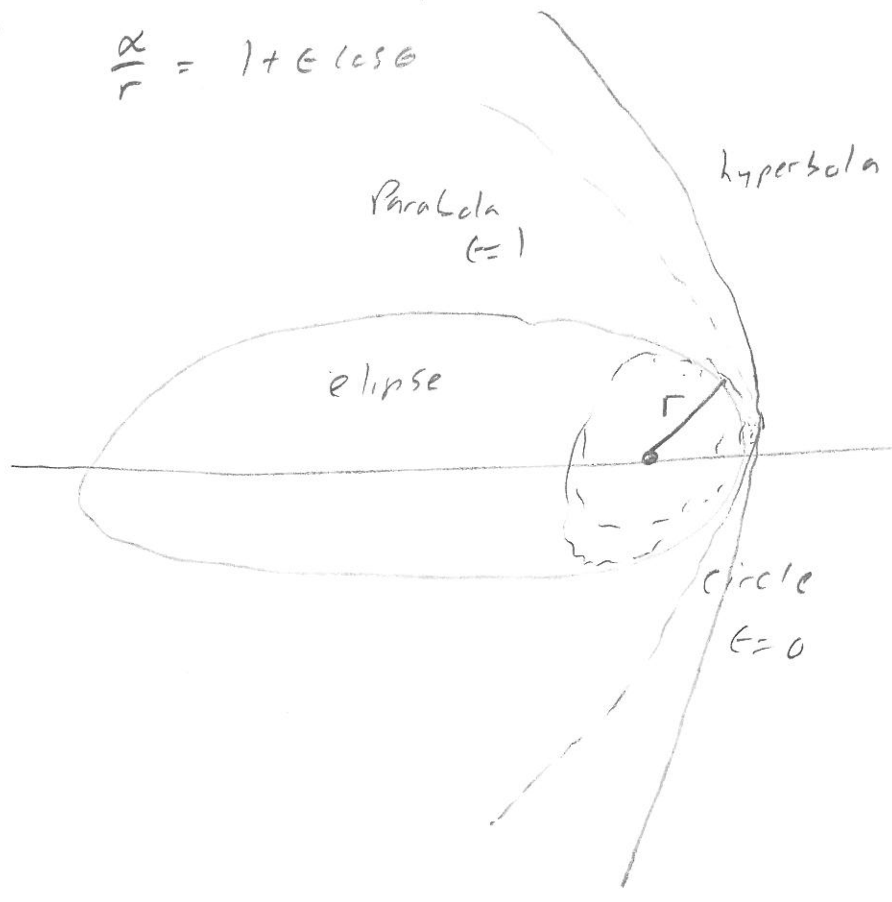
$$\theta(r) = \int \frac{\ell/r^2 \quad dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}} + c$$

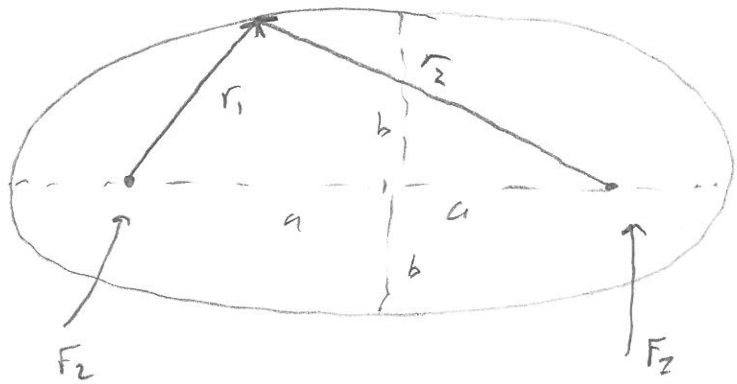
$$\cos(\theta) = \frac{\frac{\ell^2}{\mu k r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k}}}$$

where $\alpha = \frac{\ell^2}{\mu k}$

$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$

gives $\frac{\alpha}{r} = 1 + \epsilon \cos\theta$





$F_1 \to$ Foci

$r_1 + r_2 = 2a$

$r_1 + r_2 = 2a$ $\quad$ a = Semi major axis

b = Semi minor axis

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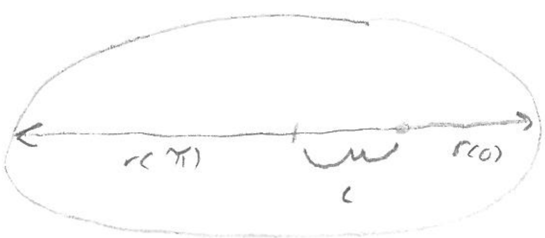
a: ?

also $\frac{\alpha}{r} = 1 + \epsilon\cos\theta$



$r(0) + r(\pi) = 2a$ $\quad$ $r = \frac{\alpha}{1+\epsilon\cos\theta} \to r_0 = \frac{\alpha}{1+\epsilon}$ $\quad$ $r(\pi) = \frac{\alpha}{1-\epsilon}$

$$r_0 + r_\pi = \frac{\alpha[1-\epsilon+1+\epsilon]}{(1+\epsilon)(1-\epsilon)} = \frac{2\alpha}{(1-\epsilon^2)}$$

$$\frac{r_0 + r_\pi}{2} = \frac{\alpha}{1-\epsilon^2} = \boxed{a = \frac{k}{2|E|}}$$

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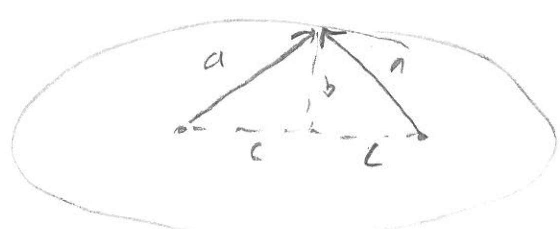
b, ?



$b = \sqrt{a^2 - c^2}$

$c = a - r(0)$

$c = a - \frac{\alpha}{1+\epsilon} = \alpha\left[\frac{1}{1-\epsilon^2} - \frac{1}{1+\epsilon}\right]$

$$\boxed{\frac{c}{\alpha} = \frac{1-(1-\epsilon)}{1-\epsilon^2} = \frac{\epsilon}{1-\epsilon^2}}$$

or $\quad b = \sqrt{\alpha a} \qquad \pi a b = \pi a^{3/2}\sqrt{\alpha}$

$$\uparrow \frac{\ell^2}{\mu k}$$

$$\tau = \frac{2\mu}{\ell}\pi a b = \frac{2\mu}{\ell}\pi a^{3/2}\sqrt{\alpha}$$

$$\tau^2 = \frac{4\mu^2}{\ell^2}\pi^2 a^3 \frac{\ell^2}{\mu k} = \frac{4\mu\pi^2 a^3}{k}$$

For $F(r) = -\frac{G m_1 m_2}{r^2} = -\frac{k}{r}$ $\qquad$ Keplers Third Law

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)} \rightarrow \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}, \quad \frac{\alpha}{r} = 1+\epsilon\cos\theta, \quad \frac{dA}{d\tau} = \frac{\ell}{2\mu}$$

$$\tau^2 = \frac{(2\pi a)^2}{\bar{v}^2}, \quad \bar{v}^2 = \frac{4\pi^2 a^2}{\tau^2} = \frac{4\pi^2 a^2}{4\pi^2 a^3}G(m_1+m_2) \rightarrow \bar{v}^2 = \frac{G(m_1+m_2)}{a}$$

$$v = \left(\frac{G(m_1+m_2)}{a}\right)^{1/2} \rightarrow r\dot{\theta} = v \qquad r \rightarrow a$$

$$\frac{v}{a} = \dot{\theta} = \left(\frac{G(m_1+m_2)}{a^3}\right)^{1/2} \qquad r\uparrow \; v\downarrow$$

$$l = \mu r^2 \dot{\theta} = \mu\left[G(m_1+m_2)a\right]^{1/2} \qquad r\uparrow \; l\uparrow$$