Hamilton's Equations

Skip 7.6, 7.7, 7.8, 7.9

One can show conditions for conservation of Energy, momentum, and angular momentum

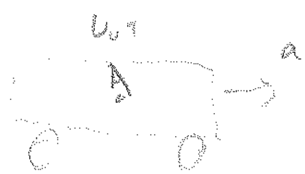
Let us define the Hamiltonian

$$H = \sum_j p_j \dot{q}_j - L$$

Iff the generalized coordinates are independent of time and U is not a function of time or velocity

Then $H = T + U = E$.

This is different than saying E = constant



$x = v_0 t + \frac{1}{2} a t^2 + \ell \sin\theta$

$\dot{x} = v_0 + at + \dot{\theta}\ell\cos\theta \qquad U = -mg\ell\cos\theta$

$y = -\ell\cos\theta$

$\dot{y} = \ell\dot{\theta}\sin\theta$

$$L = \frac{1}{2} m(v_0 + at + \ell\dot{\theta}\cos\theta)^2 + \frac{1}{2} m(\ell\dot{\theta}\sin\theta)^2 + mg\ell\cos\theta$$

$$L = \frac{1}{2} m\left[v_0^2 + a^2t^2 + \ell^2\dot{\theta}^2 + 2v_0 at + 2v_0\ell\dot{\theta}\cos\theta + 2at\ell\dot{\theta}\sin\theta\right] + mg\ell\cos\theta$$

$T + U \neq$ constant and $H \neq T + U$

Again $H = \sum_j p_j \dot{q}_j - L$

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \dot{p}_j = \frac{\partial L}{\partial q_j}}$$

go from $q_i, \dot{q}_i$ To $p_j, \dot{p}_j$

Now $H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t)$ essentially x, v_x to x, p_x

$$A \quad dH = \sum_k \left(\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dp_k \right) + \frac{\partial H}{\partial t} dt$$

$$dH = d\left[\sum_j p_j \dot{q}_j - L \right] = \sum_k \left(\underset{1}{\dot{q}_k dp_k} + \underset{2}{p_k d\dot{q}_k} - \underset{3}{\frac{\partial L}{\partial q_k} dq_k} - \underset{4}{\frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k} \right) - \frac{\partial L}{\partial t} dt$$

$$p_k d\dot{q}_k = \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \quad \text{so} \quad 2+4=0$$

$$\text{Then} \quad \sum_k \left(\overset{1}{\frac{\partial H}{\partial q_k} dq_k} + \overset{2}{\frac{\partial H}{\partial p_k} dp_k} \right) + \frac{\partial H}{\partial t} dt = \sum_k \left(\overset{3}{\dot{q}_k dp_k} - \overset{4}{\frac{\partial L}{\partial q_k} dq_k} \right) - \frac{\partial L}{\partial t} dt = dH$$

$$\frac{\partial L}{\partial q_k} = \dot{p}_k$$

Then $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$

$1 = 4$ or $\frac{\partial H}{\partial q_k} = -\dot{p}_k$ and $2 = 3$ $\frac{\partial H}{\partial p_k} = \dot{q}_k$

$$\text{and} \quad \frac{dH}{dt} = \sum_k \underbrace{\left(-\dot{p}_k \overset{\dot{q}_k}{\frac{dq_k}{dt}} + \dot{q}_k \overset{\dot{p}_k}{\frac{dp_k}{dt}} \right)}_{0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

See A

Prescription

1: $P_j = \frac{\partial L}{\partial \dot{q}_j}$

2: $H = \sum_j P_j \dot{q}_j - L$ Hamiltonian

3: $\rightarrow$ write H

4: $\boxed{\dot{q}_k = \frac{\partial H}{\partial P_k} \qquad -\dot{P}_k = \frac{\partial H}{\partial q_k}}$ Hamilton's eq's

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Parabolic Motion $\quad d\vec{s} = dx\,\hat{i} + dy\,\hat{j}$

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$

$U = mgy$

$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgy$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = P_x \qquad \frac{\partial L}{\partial \dot{y}} = P_y$

$\dot{x}P_x = m\dot{x}^2 = \frac{m^2\dot{x}^2}{m} = \frac{P_x^2}{m}$

$\dot{y}P_y = m\dot{y}^2 = \frac{m^2\dot{y}^2}{m} = \frac{P_y^2}{m}$

$H = \frac{P_x^2}{m} + \frac{P_y^2}{m} - \left(\frac{P_x^2}{2m} + \frac{P_y^2}{2m} - mgy\right) = \frac{P_x^2 + P_y^2}{2m} + mgy = T + U$

Now $\dfrac{P_x^2}{2m} + \dfrac{P_y^2}{2m} + mgy = H$

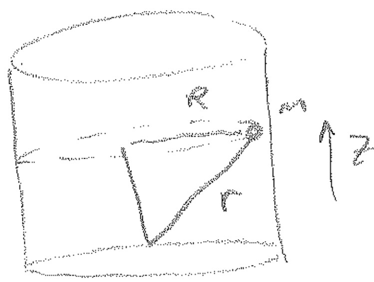
Now $\dfrac{\partial H}{\partial P_x} = \dfrac{P_x}{m} = \dot{x} \qquad \dfrac{\partial H}{\partial x} = 0 = -\dot{P}_x = -\dfrac{dP_x}{d\tau} = -\vec{F}_x = 0$

$\dfrac{\partial H}{\partial P_y} = \dfrac{P_y}{m} = \dot{y} \qquad \dfrac{\partial H}{\partial y} = mg = -\dot{P}_y = -\vec{F}_y \rightarrow \vec{F}_y = -mg$

Or $m\dfrac{dx}{d\tau} = P_x$ and $m\dfrac{dy}{d\tau} = P_y$

$\vec{F}_x = 0 \qquad \vec{F}_y = -mg$

Ex. particle on



$R^2 = x^2 + y^2$

$\vec{F} = -kr\hat{r}$

$U = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2+y^2+z^2) = \frac{1}{2}k(R^2+z^2)$

T from $ds = dr\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{z}$ $= \frac{1}{2}m(\underset{0}{\dot{R}^2} + R^2\dot{\theta}^2 + \dot{z}^2)$

$T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2)$

$L = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}k(R^2+z^2)$

$$P_\theta = \frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \qquad P_z = \frac{\partial L}{\partial \dot z} = m\dot z$$

$$\dot\theta = \frac{P_\theta}{mR^2} \qquad \dot z = \frac{P_z}{m}$$

$$P_\theta\dot\theta = \frac{P_\theta^2}{mR^2} \qquad P_z\dot z = \frac{P_z^2}{m}$$

$$H = \frac{P_\theta^2}{mR^2} + \frac{P_z^2}{m} - \left(\frac{P_\theta^2}{2mR^2} + \frac{P_z^2}{2m}\right) + \frac{1}{2}k(R^2+z^2)$$

$$H = \frac{P_\theta^2}{2mR^2} + \frac{P_z^2}{2m} + \frac{1}{2}k(R^2+z^2)$$

$$\dot\theta = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mR^2} \qquad \dot P_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot z = \frac{\partial H}{\partial P_z} = \frac{P_z}{m} \qquad \dot P_z = -\frac{\partial H}{\partial z} = -kz$$

$$P_z = m\dot z \quad \dot P_z = -kz \quad \rightarrow \quad m\ddot z = -kz \quad \ddot z + \frac{k}{m}z = 0$$

$$z(t) = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$