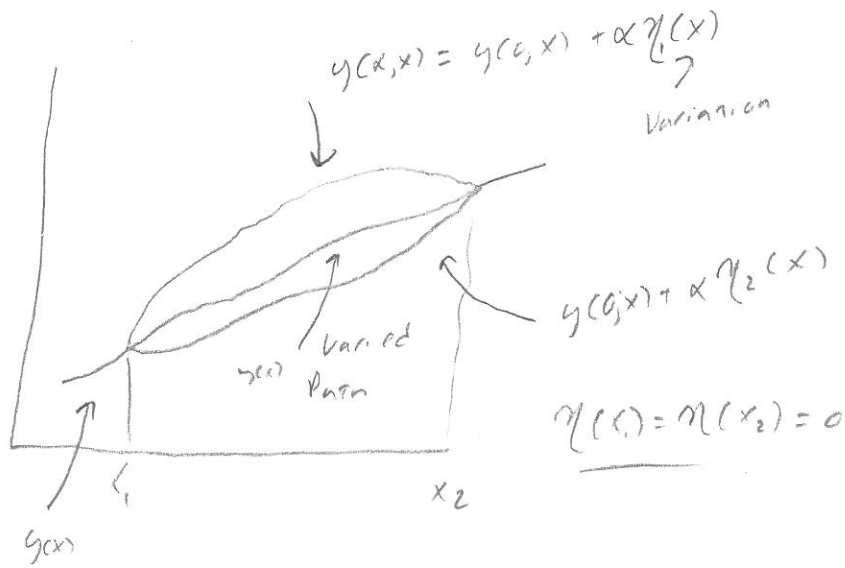


Chapter #6

Calculus of Variations

define $y(x)$ s.t. $J = \int_{x_1}^{x_2} F(y(x), y'(x), x) dx$ is an extremum (minimum)

$J \rightarrow$ Functional

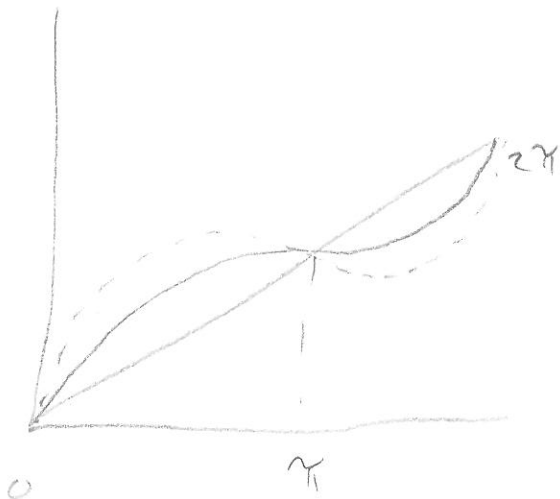


Then $J(\alpha) = \int_{x_1}^{x_2} F(y(x, \alpha), y'(x, \alpha), x) dx \rightarrow$ goal, consider all variations to $y(x)$, pick path that is an extremum

require $\left. \frac{\partial J}{\partial \alpha} \right|_{\alpha=0} = 0$

$y(x) \rightarrow$ extremum path

Ex. Consider $F = \left(\frac{dy}{dx}\right)^2$ where $y(x) = x \in [0, 2\pi]$



add $\eta(x) = \sin(x)$

So $y(x, \alpha) = x + \alpha \sin(x)$

Show minimum is $\alpha = 0$

For $J(\alpha)$

$$\frac{dy(x, \alpha)}{dx} = 1 + \alpha \cos(x)$$

$$\text{Then } F = \left(\frac{dy(x, \alpha)}{dx}\right)^2 = 1 + 2\alpha \cos(x) + \alpha^2 \cos^2(x)$$

$$J(\alpha) = \int_0^{2\pi} (1 + 2\alpha \cos(x) + \alpha^2 \cos^2(x)) dx = 2\pi + \alpha^2 \pi$$

So $\alpha \neq 0$ increases $J(\alpha)$

Euler's Equation

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha} \int_{x_1}^{x_2} F(y, y', x) dx = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} F(y, y', x) dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \alpha} \right) dx$$

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x)$$

$$y'(x, \alpha) = y'(x, 0) + \alpha \eta'(x)$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad \frac{\partial y'}{\partial \alpha} = \eta'(x)$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right] dx$$

$$\text{Now } \int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \frac{d\eta}{dx} dx = \underbrace{\frac{\partial F}{\partial y'} \eta(x)}_1 \Big|_{x_1}^{x_2} - \underbrace{\int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \eta(x) dx}_2$$

$$1 = 0 \quad \eta(x_1) = \eta(x_2) \quad 2 \neq 0$$

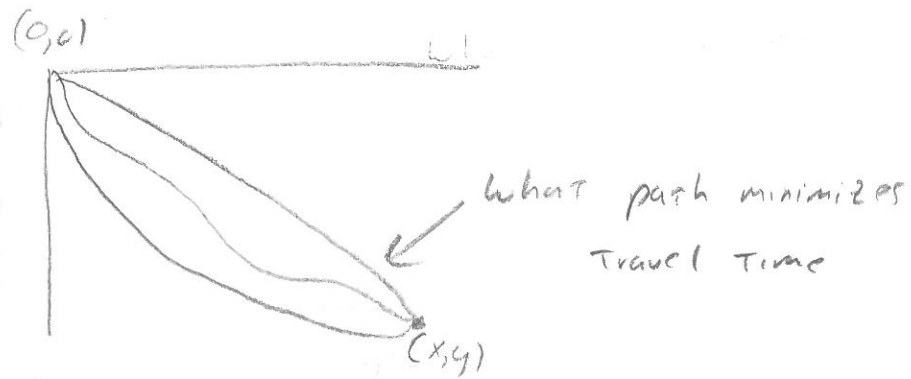
$$\text{So } \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta(x) - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \eta(x) \right) dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] \eta(x) dx$$

$$\eta(x) \rightarrow \text{any } \text{need } \frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

Euler's Equation

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Move $x, y, T(0,0)$

Now $E = T + U = 0$ if $v_{initial} = 0$

$$\text{or } 0 = -mgy + \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 \rightarrow 2gy = \left(\frac{ds}{dt}\right)^2$$

$$|\vec{v}| = \sqrt{2gy}$$

$$dt = \frac{ds}{\sqrt{2gy}} \quad \text{want to minimize } \int dt = \int \frac{ds}{\sqrt{2gy}}$$

$$ds = \sqrt{dx^2 + dy^2} = dx \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{1/2} = (1 + (y')^2)^{1/2} dx$$

$$\text{or minimize } \frac{1}{\sqrt{2g}} \int \frac{(1 + (y')^2)^{1/2} dx}{\sqrt{y}}$$

$$\gamma = J = \frac{1}{\sqrt{2y}} \int \sqrt{\frac{1+(y')^2}{y}} dx$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'}$$

$$\frac{\partial F}{\partial y} = -\sqrt{\frac{1+(y')^2}{y}} \left(\frac{1}{2y} \right)$$

$$\frac{\partial F}{\partial y'} = \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{1+(y')^2}} \cdot 2y' = \frac{y'}{\sqrt{y(1+(y')^2)}}$$

$$\frac{d}{dx} \frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y(1+(y')^2)}} \left[\frac{y''}{1+(y')^2} - \frac{1}{2} \frac{(y')^2}{y} \right] \quad \text{let } A = 1+(y')^2$$

$$\frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'} \rightarrow -\frac{A^{1/2}}{\sqrt{y}} \cdot \frac{1}{2y} = \frac{1}{A^{1/2} \sqrt{y}} \left[\frac{y''}{A} - \frac{1}{2} \frac{(y')^2}{y} \right]$$

$$-\frac{A}{2y} = \frac{y''}{A} - \frac{1}{2} \frac{(y')^2}{y} \rightarrow \frac{(y')^2 - A}{2y} = \frac{y''}{A}$$

$$-\frac{1}{2y} = \frac{y''}{A} \rightarrow \boxed{\frac{y''}{A} = -\frac{1+(y')^2}{2y}}$$

$$\boxed{1 + 2y(y'') + (y')^2 = 0}$$

$$\text{Now } 1+(y')^2 - 2yy'' = 0$$

$$\text{Let } y' = u \quad y'' = \frac{du}{dx} = \frac{du}{dy} y' = u \frac{du}{dy} \quad \text{So } 2yy'' = 2yu \frac{du}{dy}$$

$$1 + u^2 + 2yu \frac{du}{dy} = 0$$

$$(1+u^2) + y \frac{2u du}{dy} = 0$$

$$\frac{dy}{y} + \frac{2u du}{(1+u^2)} = 0$$

$$\ln(y) + \ln(1+u^2) = 0$$

$$y(1+u^2) = D \rightarrow \frac{D-y}{y} = u^2$$

$$u = \left(\frac{D-y}{y} \right)^{1/2} = \frac{dy}{dx}$$

$$\text{Now } \int \sqrt{\frac{y}{D-y}} dy = \int dx$$

$$x + C = \int \sqrt{\frac{y}{D-y}} dy \quad \text{Let } y = D \sin^2 \theta$$

$$x+c = \int \sqrt{\frac{D \sin^2 \theta}{D \cos^2 \theta}} \cdot 2D \sin \theta \cos \theta d\theta$$

$$x+c = 2D \int \sin^2 \theta d\theta = 2D \left(\frac{\theta}{2} - \frac{\cos(2\theta)}{2} \right) d\theta$$

$$x+c = \frac{1}{2} D [2\theta - \sin(2\theta)]$$

$$x = \frac{1}{2} D [2\theta - \sin(2\theta)] - c$$

$$y = D \sin^2 \theta = \frac{D}{2} (1 - \cos(2\theta))$$

Starts at $c, 0 \rightarrow c=0$

$$x = \frac{D}{2} (2\theta - \sin(2\theta)) \quad y = \frac{D}{2} (1 - \cos(2\theta))$$

$$\frac{D}{2} = A \quad 2\theta = \phi$$

$$x = A [\phi - \sin(\phi)] \quad y = A [1 - \cos(\phi)]$$

