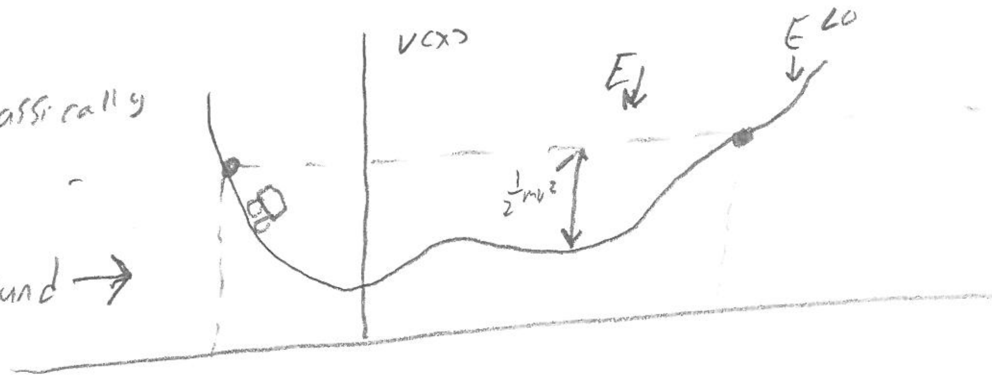
# DELTA FUNCTION AND POTENTIAL

## 1st Bound and Scattering States

Classically

Bound →



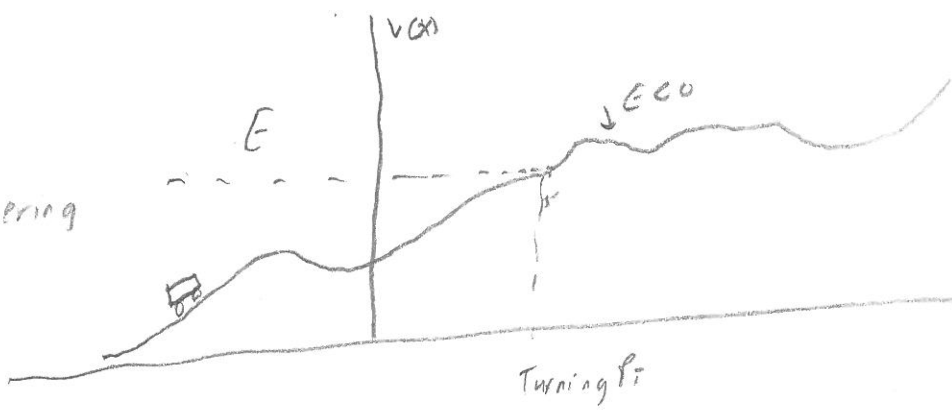

$E = \frac{1}{2}mv^2 + V(x)$

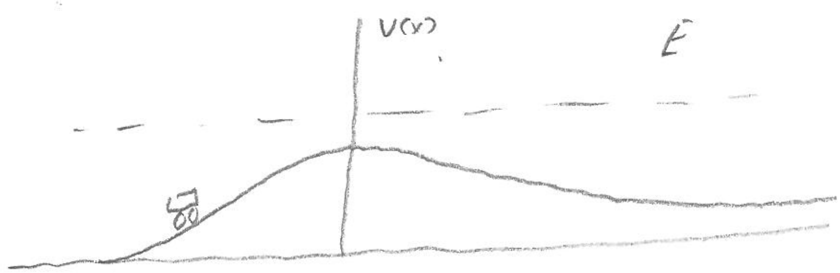
↑ positive or negative

$E - V(x) = \frac{1}{2}mv^2$

car, orbit, etc

Scattering



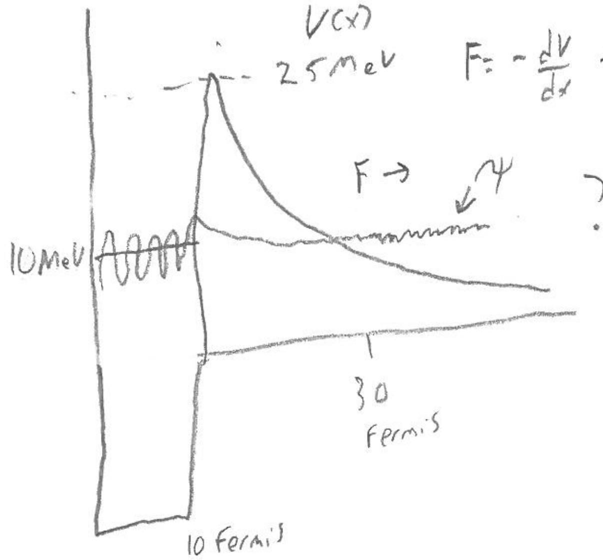




For $E<0$ bound

$E>0$ Scattering

but → in QM Finite chance For Tunneling

$F = -\frac{dV}{dx}$ → Slope Tangentline, (-1)

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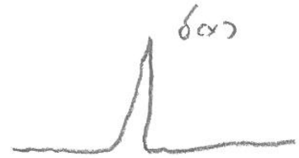
Delta Function $\delta(x) = 0 \quad x \neq 0$
$\qquad\qquad\qquad\quad \infty \quad x = 0$




$$\int \delta(x)\, dx = 0 \cdot \infty = 1$$

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$\delta(x-a) \rightarrow \quad 0 \quad x \neq a$
$\qquad\qquad\quad\;\; \infty \quad x = a$

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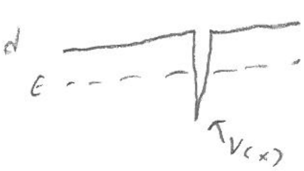
$F(x)\,\delta(x-a) = F(a)\,\delta(x-a$

$$\int F(x)\,\delta(x-a)\, dx = F(a)$$

Consider $V(x) = -\alpha\,\delta(x)$ with $E < 0$ Bound

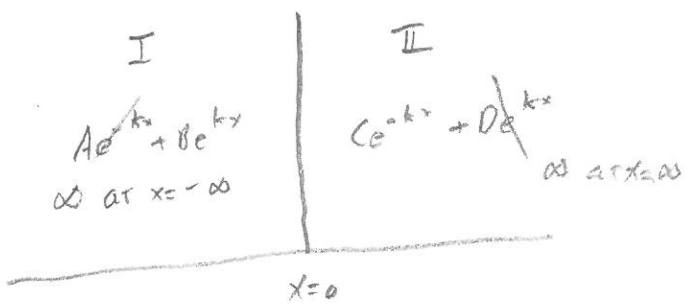



$$\hat{H}\psi = E\psi \rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x) = E\psi$$

Be careful $E <$ or $> 0$
$< 0 \rightarrow$ Bound

Or $\psi'' = \frac{-2mE\psi}{\hbar^2} = k^2\psi$ $\quad k = \frac{\sqrt{-2mE}}{\hbar}$ ↖ positive since $E<0$

$\psi(x) = Ae^{-kx} + Be^{kx}$ NOT WAVES Find acceptable Solutions

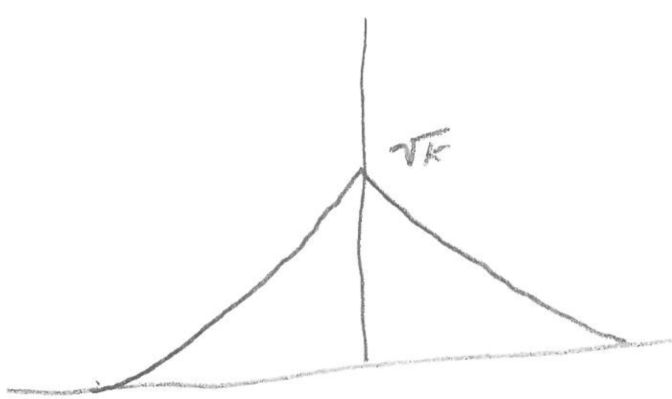



1. $\psi$ is continuous
2. $\psi'$ is continuous EXCEPT where $V = \pm\infty$

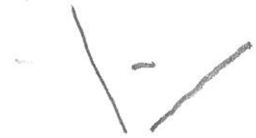
$Be^{kx}\Big|_{x=0} = Ce^{-kx}\Big|_{x=0}$ $\quad B = C$

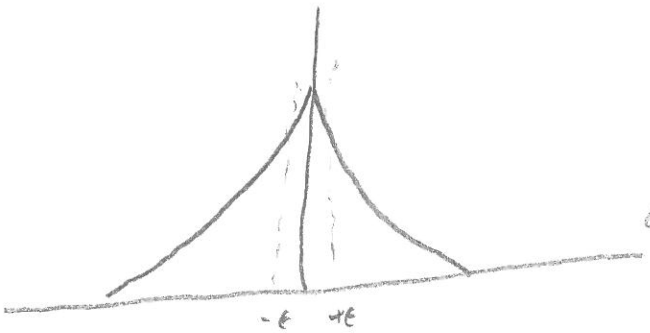
$\psi(x) = \begin{cases} Be^{kx} & x \le 0 \\ Be^{-kx} & x \ge 0 \end{cases}$ $\rightarrow$ $2B^2\int_0^\infty e^{-2kx}\,dx = 1$ $\quad \frac{2B^2}{2k}\int_{-\infty}^0 e^u\,du$

$\frac{B^2}{k} = 1 \quad B = \sqrt{k}$



But, $\psi'\big|_{x=0}$ is discontinuous






$\lim \epsilon \to 0$

Integrate $\lim_{\epsilon \to 0} \left[ -\frac{\hbar^2}{2m} \int_{-\epsilon}^{+\epsilon} \frac{d^2\psi}{dx^2} + \int_{-\epsilon}^{+\epsilon} V(x)\psi = E \int_{-\epsilon}^{+\epsilon} \psi \, dx \right]$

$\psi(0) - \psi(0) = 0$

$$\Delta\left(\frac{d\psi}{dx}\right) = \lim_{\epsilon \to 0} \left( \frac{d\psi}{dx}\bigg|_{\epsilon} - \frac{d\psi}{dx}\bigg|_{-\epsilon} \right) = \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} V(x)\psi \, dx$$

$$\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m}{\hbar^2} \int_{-\epsilon}^{+\epsilon} -\alpha\,\delta(x)\psi \, dx \quad -\frac{2m\alpha}{\hbar^2}\psi(0)$$

but $\frac{d\psi}{dx}\bigg|_{+} = -Bke^{-kx}\bigg|_{x=0} = -Bk$

right

$\frac{d\psi}{dx}\bigg|_{-} = Bke^{kx}\bigg|_{x=0} = Bk$

Left

$\psi(0)$

and $\Delta\left(\frac{d\psi}{dx}\right) = -Bk - (Bk) = -2Bk = -\frac{2m\alpha}{\hbar^2} B$

So $k = \frac{m\alpha}{\hbar^2}$

$k = \frac{-m\alpha}{\hbar^2} = \frac{\sqrt{-2mE}}{\hbar}$ $\qquad \frac{m^2\alpha^2}{\hbar^4} = \frac{-2mE}{\hbar^2}$ $\qquad \boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$ **Bound**

**One State**

Now $B = \sqrt{k} = \frac{\sqrt{m\alpha}}{\hbar}$ and $\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}}$

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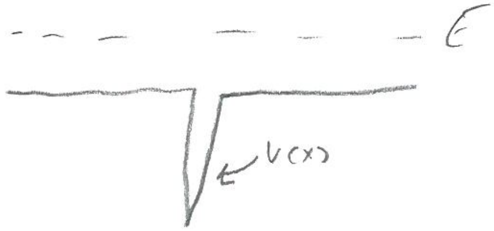
Scattering




$\psi'' = \frac{-2mE}{\hbar^2} = -k^2\psi \qquad k = \frac{\sqrt{2mE}}{\hbar}$ $\qquad x>0$, $x<0$ Not $x=0$ V exists there

| I | II |
|---|---|
| $\overset{\rightarrow}{Ae^{ikx}} + \overset{\leftarrow}{Be^{-ikx}}$ | $\overset{\rightarrow}{Ce^{ikx}} + \overset{\leftarrow}{De^{-ikx}}$ |

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$\psi(x=0)$ continuous $\qquad A+B = C+D \qquad$ 1 equation

$$\left.\frac{d\psi}{dx}\right|_{+,\,x=0} = ik(C-D) \qquad \left.\frac{d\psi}{dx}\right|_{-} = ik(A-B)$$

As before integrating $\int_{-\epsilon}^{+\epsilon} [\hat{H}\psi = E\psi]$ $\lim_{\epsilon \to 0}$

gives $\Delta\left(\frac{d\psi}{dx}\right) = \frac{-2m\alpha}{\hbar^2}\psi(0) = \frac{-2m\alpha}{\hbar^2}(A+B)$
↑ pick one

So $ik(C-D+B-A) = \frac{-2m\alpha}{\hbar^2}(A+B)$

Now D represents a leftward ray from the right

It doesn't exist if we start from the left, nothing to scatter from at $x=\infty$

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So $A+B=C$, $ik(C+B-A) = -\frac{2m\alpha}{\hbar^2}(A+B)$

Uh oh Two eqs 3 unknowns Can't solve, can find reflected vs transmitted though

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A= incident B= reflected C= Transmitted

Solve in terms of A → incident

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Let $\beta = \frac{m\alpha}{\hbar^2 k}$ Then $C+B-A = \frac{-2m\alpha}{i k\hbar^2}(A+B) = \frac{2m\alpha i}{\hbar^2 k}(A+B) = 2\beta i(A+B)$

$C = A - B + 2\beta i(A+B) = (1+2\beta i)A - (1-2\beta i)B$

Now $A+B=C$ , $(1+2\beta i)A-(1-2\beta i)B=C$

$$(1+2\beta i)A-(1-2\beta i)B=A+B$$

$$(1+2\beta i)A-A=B+(1-2\beta i)B$$

$$A[2\beta i]=2(1-\beta i)B \rightarrow B=\frac{A(\beta i)}{1-\beta i} \qquad B^*B=\frac{|A|^2(-\beta i)(-\beta i)}{(1-\beta i)(1+\beta i)}$$

$$\boxed{\frac{|B|^2}{|A^2|}=\frac{\beta^2}{1+\beta^2}}$$

Reflection $R\rightarrow$ % incident reflected

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$$B=\frac{A(\beta i)}{1-\beta i} \qquad A+B=C \qquad A+\frac{A(\beta i)}{1-\beta i}=A\left(1+\frac{\beta i}{1-\beta i}\right)=\frac{1-\beta i+\beta i}{1-\beta i}A=C$$

$$\frac{A}{1-\beta i}=C \qquad \frac{|A|^2}{(1-\beta i)(1+\beta i)}=|C|^2$$

$$\frac{|C|^2}{|A|^2}=\frac{1}{1+\beta^2}$$ $\rightarrow$ Transmitted $T\rightarrow$ % Transmitted

$$R+T=\frac{\beta^2}{1+\beta^2}+\frac{1}{1+\beta^2}=\frac{1+\beta^2}{1+\beta^2}=\underline{\underline{1}}$$

$$R=\frac{1}{1+(2\hbar^2E/m\alpha^2)} \qquad T=\frac{1}{1+m\alpha^2/2\hbar^2E}$$

Weird $E>V$ Should $=0$, doesn't