Stationary States + ISW

SE = $i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$

SOV → $\Psi(x,t) = \psi(x)\phi(t)$? Try it!

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$\frac{\partial \Psi}{\partial t} = \psi \frac{\partial \phi}{\partial t} \qquad \frac{\partial^2 \Psi}{\partial x^2} = \phi \frac{\partial^2 \psi}{\partial x^2}$$

or $i\hbar \psi \frac{\partial \phi}{\partial t} = \frac{-\hbar^2}{2m}\phi\frac{\partial^2 \psi}{\partial x^2} + \phi\psi V$ divide by $\phi\psi$

$i\hbar \frac{1}{\phi}\frac{\partial \phi}{\partial t} = \frac{-\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} + V$ $F(t) = G(x)$? only if $C = C$
call $C = E$

constant on each side

$i\hbar \frac{1}{\phi}\frac{\partial \phi}{\partial t} = E$ → # → Energy

$\boxed{\frac{\partial \phi}{\partial t} = \frac{\phi E}{i\hbar} = -\frac{\phi i E}{\hbar}}$ and $\frac{-\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2 \psi}{\partial x^2} + V = E$

or

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{2m}{\hbar^2}V\psi = -\frac{2m}{\hbar^2}E\psi$$

or

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2}(V-E)\psi}$$

1 PDE $\rightarrow$ 2 ODES

$$\frac{d\phi}{dt} = -\frac{iE}{\hbar}\phi \rightarrow \frac{d\phi}{\phi} = -\frac{iE}{\hbar}dt \rightarrow \ln(\phi) = -\frac{iEt}{\hbar} + C$$

$$\rightarrow \boxed{\phi = e^{-iEt/\hbar + C} = \tilde{A}e^{-itE/\hbar}}$$

Time Independen SE $\rightarrow \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V-E)\psi$

Solution $\rightarrow = \psi(x) \rightarrow \boxed{\Psi(x,t) = \psi(x)e^{-iEt/\hbar}}$

1.) Stationary States $\rightarrow |\Psi(x,t)|^2 = \psi^*(x)e^{iEt/\hbar}\psi(x)e^{-iEt/\hbar}$

$= |\psi^*\psi|$

$\langle Q(x,p)\rangle = F(x)$ only. Time independent

$\frac{d\langle x\rangle}{dt} = 0 \quad \langle p\rangle = 0$

Not True if $\Psi(x,t) = \sum \psi_i(x)\phi_i(t)$

$= C_1\psi_1 e^{-iE_1t/\hbar} + C_2\psi_2 e^{-iE_2t/\hbar} + \ldots\ldots$

2.) States of definite Total Energy

$$H(x,p) = \frac{p^2}{2m} + V(x)$$

$\uparrow$ no T dependence

$$\boxed{\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V}$$

Time independent SE $\hat{H}\psi = E\psi$

$\uparrow$ operator $\uparrow$ number

RULE!!

$$\langle \hat{H} \rangle = \int \psi^* \hat{H} \psi \, dx = \int \psi^* \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V \right) \psi \, dx = E \underbrace{\int |\psi|^2 dx}_{1} = E$$

$\langle \hat{H}^2 \rangle$? $\hat{H}\hat{H}\psi = \hat{H}E\psi = E\hat{H}\psi = E^2\psi$

$$\langle \hat{H}^2 \rangle = \int \psi^* \hat{H}^2 \psi \, dx = E^2 \underbrace{\int |\psi|^2 dx}_{1} = E^2$$

$$\sigma_H^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 = E^2 - E^2 = 0$$

SEPARABLE Solutions have no variance in E

3.) General Solutions $\sum \psi_i e^{-iE_i t/\hbar}$

Each state has distinct allowed energy

$$\Psi(x,t) = \sum c_n \psi_n e^{iE_n t/\hbar}$$

General Solution is Sum of Stationary Solutions

Stationary have Time independence, general DO NOT!

Job → #1 Specify $V(x)$

#2 from Solution find $\Psi(x,0) = \sum c_n \psi_n(x)$

I'll show you how later

#3 find c_n's from normalization

c_n Tell's "how much" of each ψ_n is in Solution

$\sum |c_n|^2 = 1$

$|c_n|^2$ Tells likelihood of measuring State associated with ψ_n

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Ex $\Psi(x,0) = c_1 \psi_1(x) + c_2 \psi_2(x)$

$\Psi(x,t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2(x) e^{-iE_2 t/\hbar}$

New $\int \psi_m^* \psi_n \, dx = \delta_{mn} \rightarrow = \begin{cases} 1 & m=n \\ 0 & \text{else} \end{cases}$

$$|\Psi(x,t)|^2 = (c_1^* \psi_1^* e^{iE_1 t/\hbar} + c_2^* \psi_2^* e^{iE_2 t/\hbar}) \cdot (c_1 \psi_1 e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2 t/\hbar})$$

$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + c_1^* c_2 \psi_1^* \psi_2 e^{i(E_1 - E_2)t/\hbar} + c_2^* c_1 \psi_2^* \psi_1 e^{-i(E_1 - E_2)t/\hbar}$$

$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2 c_1 c_2 \psi_1 \psi_2 \cos(E_1 - E_2)t/\hbar$$

Temporal Motion with $\frac{E_1 - E_2}{\hbar} = \omega$ ↑ frequency

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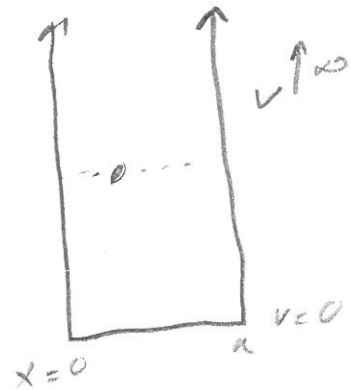
Infinite Square Well

$V(x) = \begin{cases} 0 & x \in [0, a] \\ \infty & \text{else} \end{cases}$



$\hat{H}\psi = E\psi$ in well $V = 0 \rightarrow -\frac{\hbar^2}{2m}\psi'' = E\psi$

$\psi'' = -\frac{2mE}{\hbar^2}\psi = -k^2\psi \quad k = \sqrt{\frac{2mE}{\hbar^2}}$

Now $A\psi'' + B\psi' + C = 0$ can be factored into (A, B, C can be $> < 0$ real, or imaginary)

$\psi'' + \frac{B}{A}\psi' + \frac{C}{A} = 0$ or $\frac{d}{dx} \rightarrow \hat{D} \rightarrow \hat{D}^2\psi + E\hat{D}\psi + F = 0$

or $(D + G)(D + F)\psi = 0$

$(D + 2)(D - 2)\psi = (D^2 - 4)\psi$ Solutions

$\frac{d}{dx}\psi \pm 2\psi = 0 \quad \frac{d\psi}{\psi} = \mp 2dx \rightarrow Ae^{-2x} + Be^{2x}$

$(D + 2i)(D - 2i) = (D^2 + 4)\psi$

$\psi = Ae^{2ix} + Be^{-2ix}$

back to $\psi'' = -k^2\psi$

$$D^2\psi + k^2\psi = 0 \rightarrow (D+ik)(D-ik)\psi = 0$$

$$\psi = Ae^{ikx} + Be^{-ikx}$$

can be written as $\tilde{C}\cos(kx) + \tilde{D}\sin(kx)$

Algorithm $\psi(x) \rightarrow 0$ as $x \rightarrow \pm\infty$

$\psi, \frac{d\psi}{dx}$ continuous

at $x=0, x=a$ $V=\infty$, ψ dies

or $\psi(0) = \psi(a) = 0$

$$\psi(0) = \underset{\neq 0}{C\cos(0)} + D\underset{\uparrow 0}{\sin(0)} = 0 \quad \text{No}$$

$\underline{\underline{C=0}}$

Now $D\sin(ka) = 0$ only possible if

$ka = n\pi \rightarrow k = \pm\frac{n\pi}{a}$ but $-\sin(x) = \sin(-x)$

so absorb $-$ into D

We've just $\underline{\underline{\text{quantized}}}$ E

remember $p_n = \hbar k_n = \frac{\hbar n \pi}{a}$

$$\boxed{E_n = \frac{p^2}{2m} = \left(\frac{\hbar n \pi}{a}\right)^2 \cdot \frac{1}{2m}}$$

Now $\psi_n(x) = A\,SIN(k_n x)$

$A = ?$ $\int_0^a A^2 SIN^2(k_n x)\,dx = 1$ $u = k_n x$ $du = k_n dx$

$x = 0$ $u = 0$

$x = a$ $u = n\pi$

$$\frac{A^2}{k_n}\int_0^{n\pi} \frac{1 - \cos(2u)}{2}\,dx = \frac{A^2}{k_n}\left[\int \frac{1}{2}\,du - \int \underbrace{\frac{\cos(2u)}{2}\,du}_{0}\right]$$

$$= \frac{A^2 \cdot n\pi}{2 k_n} = \frac{A^2 n \pi}{2} \cdot \frac{a}{n\pi} = 1 = \frac{A^2 a}{2}$$

$$\boxed{A = \sqrt{\frac{2}{a}}}$$

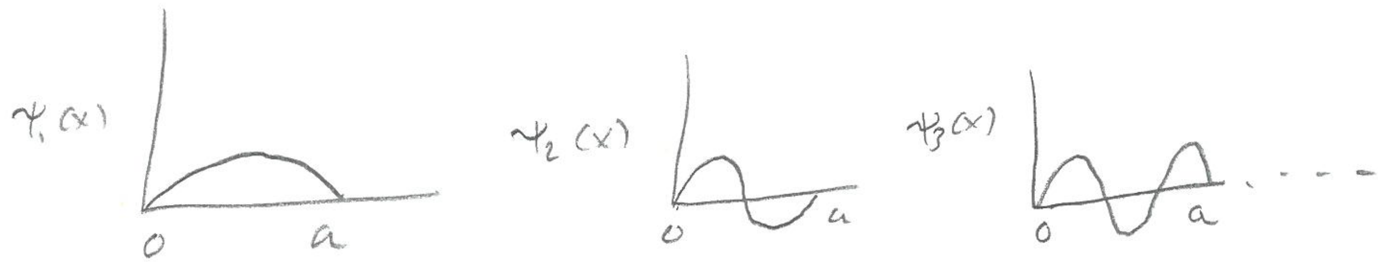
$$\boxed{\text{Now } \Psi(x,t) = \sum \sqrt{\frac{2}{a}}\,SIN\left(\frac{n\pi}{a}x\right) e^{-i\left(\frac{\hbar n \pi}{a}\right)^2 \cdot \frac{t}{2m\hbar}}}$$

1.) Even → odd → Even → odd with n

n=0? doesn't exist

n=1 even n=2 odd . . . . . .

1 node $n=1$ 2 nodes $n=2$ . . . . . .



$E\uparrow$

Orthogonal $\rightarrow$ if $m=n$ $\int A^2 SIN^2\left(\frac{n\pi}{a}x\right) dx = 1$

if $m \neq n$ $\int \psi_m^* \psi_n \, dx = 0$

See Page 33

USE THIS!!!

$$\int \psi_m^* \psi_n \, dx = \delta_{mn}$$

$$F(x) = \sum \frac{x^n}{n!} \frac{d^n F}{dx^n} \quad \text{or} \quad F(x) = \sum C_n \psi_n = \sqrt{\frac{2}{a}} \sum_n C_n \, SIN\left(\frac{n\pi x}{a}\right)$$

Complete! Any function can be written as a linear combination

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How?

C_n? $\int \psi_m^* f(x) = \sum C_n \int \psi_m^*(x)\psi_n(x)\,dx$

$= \sum C_n \delta_{mn} = C_m$

$C_n = \int \psi_n^*(x)\, f(x)\, dx$

and

$$\Psi(x,t) = \sum C_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) e^{-i\left(\frac{n^2\pi^2\hbar}{2ma^2}\right)t}$$

$$\Psi(x,0) = \sum C_n \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \rightarrow C_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{a}\right)\Psi(x,0)\,dx$$

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1. To get $\Psi(x,t)$ use form for C_n

Example $\Psi(x,0) = Ax(a-x)$ general Expansion?

$$\int_0^a |\Psi(x,0)|^2 dx = \int_0^a A^2 x^2 (a-x)^2 dx = 1$$

$$\int_0^a A^2 x^2 (a^2 + x^2 - 2ax)\,dx = 1 \quad A^2\left[\frac{x^3 a^2}{3} + \frac{x^5}{5} - \frac{2a x^4}{4}\right]\Big|_0^a = 1$$

$$A^2\left[\frac{a^5}{3} + \frac{a^5}{5} - \frac{a^5}{2}\right] = 1$$

$$A^2\left(\frac{10 + 6 - 15}{30} a^5\right) = 1 \quad \frac{a^5}{30}A^2 = 1 \quad \underline{\underline{A = \sqrt{\frac{30}{a^5}}}}$$

$$F(x) = \sum c_n \psi_n(x)$$

$$\int \psi_m^* F(x)\, dx = \sum c_n \int \psi_m^* \psi_n \, dx$$

$$= \delta_{mn} \sum c_n$$

$$c_n = \int_0^a \underbrace{\sqrt{\frac{2}{a}} \cdot \sin\left(\frac{m\pi x}{a}\right)}_{\psi_m^*} \underbrace{A\,(x)(a-x)\, dx}_{F(x)\, dx}$$

$$= \sqrt{\frac{2}{a}} \cdot \left(\frac{30}{a^5}\right)^{1/2} \int_0^a \sin\left(\frac{m\pi x}{a}\right)(xa - x^2)\, dx$$

Integrate Away! . . . . generally, use integrator

$$= \frac{4\sqrt{15}}{(n\pi)^3}\Big[\underset{1}{\cos(0)} - \underset{-1,\,1,\,-1,\,\ldots}{\cos(n\pi)}\Big]$$

$$\boxed{= \begin{cases} 0 & n \text{ even} \\ \dfrac{8\sqrt{15}}{(n\pi)^3} & n \text{ odd} \end{cases}}$$

$$\Psi(x,t) = \sqrt{\frac{30}{a}}\left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin\left(\frac{n\pi x}{a}\right) e^{-iE_n t/\hbar}$$

Generally $|c_n|^2 =$ P of measuring state ψ_n

$$\sum |c_n|^2 = 1$$

For $\Psi(x,0) = A\,x(a-x)$

$P(\Psi_1) = |c_1|^2 = \frac{8\sqrt{15}}{\pi^3}$

$P(\Psi_2) = |c_2|^2 = 0$

$P(\Psi_3) = |c_3|^2 = \frac{\cancel{8}\sqrt{15}}{\cancel{8}\pi^3}$

$\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n \rightarrow$ just $\sum P(n) E_n$

$H\Psi_n = E_n \Psi_n$

$\langle H \rangle = \int \Psi^* \hat{H} \Psi \, dx = \int \left(\sum c_m \Psi_m\right)^* H \sum c_n \Psi_n \, dx$

$= \underbrace{\sum\sum c_m^* c_n E_n}\int \Psi_m^* \Psi_n \, dx = \sum |c_n|^2 E_n$

δ_{mn}

$\left[\left(\underset{1}{c_1^* c_1} + \underset{0}{c_1^* c_2} + \underset{0}{c_1^* c_3} + \dots\right) + \left(\underset{0}{c_2^* c_1} + \underset{1}{c_2^* c_2} + \underset{0}{c_2^* c_3} + \dots\right) + \dots\dots\right]\delta_{mn}$

USE THIS

Let's say $\psi(x) = \frac{1}{\sqrt{2}} \psi_2(x) + \frac{1}{\sqrt{4}} \psi_5(x) + \frac{1}{\sqrt{4}} \psi_7(x)$

$$\sum |c_n|^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$$

$$P(\psi_5) = \left(\frac{1}{\sqrt{4}}\right)^2 = \frac{1}{4}$$

measure 100 times $P(\psi_2) = \frac{1}{2}$ # times $= 100 P(\psi_2) = 50$