Notes 2

Normalizable means $\psi(\pm\infty) \to 0$

Normalization

#1. get ψ → Ex $Ae^{-\lambda|x|}e^{-i\omega t}$

#2. Take $|\psi|^2 = \psi\psi^*$ → Ex $(Ae^{-\lambda|x|}e^{-i\omega t})(Ae^{-\lambda|x|}e^{i\omega t})$

$$A^2e^{-2\lambda|x|}e^{i\omega t - i\omega t} = A^2e^{-2\lambda x}$$

#3. Make $\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$ Ex $\int_{-\infty}^{+\infty} A^2e^{-2\lambda|x|}dx = 1$

$$2A^2\int_0^{\infty} e^{-2\lambda x}dx = 1 \qquad u = -2\lambda x \quad -\frac{du}{2\lambda} = dx \qquad x=\infty \;\; u=-\infty \qquad x=0 \;\; u=0$$

$$-\frac{2A^2}{2\lambda}\int_0^{-\infty} e^u du = 1 \qquad -\frac{2A^2}{2\lambda}\Big|(e^{-\infty} - 1) = 1 \to \frac{A^2}{\lambda} = 1 \quad A = \sqrt{\lambda}$$

$$\psi(x,t) = \sqrt{\lambda}\,e^{-\lambda|x|}e^{-i\omega t}$$

Once Normalized

$$\int_a^b |\psi|^2 dx = P_\psi \text{ in } [a,b]$$

$$\int_{-\infty}^{+\infty} x|\psi|^2 dx = \langle x \rangle$$

$$\int_{-\infty}^{+\infty} f(x)\, |\Psi|^2 dx = \langle f(x) \rangle$$

Remember measuring one particle returns a state.

$\langle\ \rangle$ is the average of repeated measurements on identically prepared systems.

$\langle x \rangle \rightarrow$ Expectation value of location $\rightarrow$ where, on average, will we find a particle with a given state?

$\langle p \rangle \rightarrow$ " " momentum

Ex $\Psi = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \rightarrow \langle x \rangle = \int_0^a \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right) dx$

$$\frac{2}{a} \int_0^a x \sin^2\left(\frac{\pi x}{a}\right) dx = \frac{2}{a} \int_0^a \frac{x}{2}\left(1 - \cos\left(\frac{2\pi x}{a}\right)\right) dx$$

$$= \frac{1}{a} \int_0^a \left(x - x\cos(2\pi x/a)\right) dx = \frac{1}{a}\left[\frac{x^2}{2}\Big|_0^a - \int_0^a x\cos\left(\frac{2\pi x}{a}\right) dx\right]$$

$u = \frac{2\pi x}{a}$ $\quad du = \frac{2\pi dx}{a}$

$$= \frac{1}{a}\left[\frac{a^2}{2} - \left(\frac{a}{2\pi}\right)^2 \int_0^{2\pi} u\cos(u)\, du\right]$$

$u\sin(u) + \cos(u)$

$\downarrow 0 \qquad \downarrow 0$

$$= \boxed{\frac{a}{2}}$$

Now, if $\Psi = \Psi(t)$ what happens to $\langle x \rangle$?

$$\frac{d\langle x\rangle}{dt} = ?$$ back up

$|\Psi(x,t)|^2$ represents probability density for finding particle at time t in point x.

Does $\int |\Psi|^2 dx = 1$ stay true? Once normalized does it stay normalized?

$$\frac{d}{dt}\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}|\Psi(x,t)|^2 dx$$

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{\partial}{\partial t}(\Psi\Psi^*) = \Psi^*\frac{\partial \Psi}{\partial t} + \Psi\frac{\partial \Psi^*}{\partial t}$$

but SE $\rightarrow$ $i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \rightarrow \frac{\partial \Psi}{\partial t} = \frac{-\hbar}{i2m}\Psi'' + \frac{V}{i\hbar}\Psi$

$$\boxed{\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\Psi'' - \frac{iV}{\hbar}\Psi \quad \text{and} \quad \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar\Psi^{*''}}{2m} + \frac{iV}{\hbar}\Psi}$$

USE

$$\frac{\partial}{\partial t}|\psi|^2 = \psi^*\left[\frac{i\hbar}{2m}\psi'' - \frac{i}{\hbar}V\psi\right] + \psi\left[-\frac{i\hbar}{2m}(\psi^*)'' + \frac{i}{\hbar}V\psi^*\right]$$

$$= \frac{i\hbar}{2m}\left[\psi^*\psi'' - \psi(\psi^*)''\right]$$

$$= \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi^*\psi' - (\psi^*)'\psi\right]$$

result

Now $\boxed{\frac{d}{dt}\int_{-\infty}^{+\infty}|\psi(x,t)|^2\,dx = \frac{i\hbar}{2m}\int\frac{\partial}{\partial x}\left[\psi^*\psi' - (\psi^*)'\psi\right]dx}$

$$= \frac{i\hbar}{2m}\left[\psi^*\psi' - (\psi^*)'\psi\right]\Big|_{-\infty}^{+\infty} = 0$$

$\boxed{\frac{d}{dt}\int_{-\infty}^{+\infty}|\psi(x,t)|^2\,dx = 0 \quad \text{Normalization is conserved}}$

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back to $\frac{d\langle x\rangle}{dt} = \int x\frac{\partial}{\partial t}|\psi|^2\,dx = \frac{i\hbar}{2m}\int x\frac{\partial}{\partial x}\left[\psi^*\psi' - (\psi^*)'\psi\right]dx$

$d(uv) = u\,dv + v\,du \rightarrow \quad d(uv) - v\,du = u\,dv$

$$uv\Big|_a^b - \int_a^b v\,du = \int_a^b u\,dv$$

$x = u \quad \frac{\partial}{\partial x}[\psi^*\psi' - (\psi^*)'\psi]\,dx = dv$

$dx = du \quad [\psi^*\psi' - (\psi^*)'\psi] = v$

$$\frac{d\langle x\rangle}{dt} = \frac{i\hbar}{2m}\left[\underbrace{x[\psi^*\psi' - (\psi^*)'\psi]\Big|_{-\infty}^{+\infty}}_{0} - \int(\psi^*\psi' - (\psi^*)'\psi)\,dx\right]$$

$$= -\frac{i\hbar}{2m}\int(\psi^*\psi' - (\psi^*)'\psi)\,dx$$

$$= -\frac{i\hbar}{2m}\int\psi^*\psi'\,dx + \frac{i\hbar}{2m}\int(\psi^*)'\psi\,dx$$

$\rightarrow \frac{\partial\psi^*}{\partial x}dx = dv \quad u = \psi$

$\psi^* = v \quad du = \frac{\partial\psi}{\partial x}$

$$\frac{i\hbar}{2m}\left[\underbrace{\cancel{\psi\psi^*}\Big|_{-\infty}^{+\infty}}_{0} - \int\psi^*\frac{\partial\psi}{\partial x}dx\right]$$

$$\boxed{= -\frac{i\hbar}{m}\int\psi^*\frac{\partial\psi}{\partial x}dx}$$

?

$\langle v\rangle = \frac{d\langle x\rangle}{dt} \quad \langle p\rangle = m\frac{d\langle x\rangle}{dt} = i\hbar\int\left(\psi^*\frac{\partial\psi}{\partial x}\right)dx$

Operators $\hat{Q}$

$\hat{x} = x$ $\quad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ or $-i\hbar\frac{\partial}{\partial x}$

$\langle x \rangle = \int \Psi^*(x)\Psi \, dx$

$\langle p \rangle = \int \Psi^* \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi \, dx$

Recipe $\langle Q \rangle = \int \Psi^* \hat{Q} \Psi \, dx$

all $\hat{Q} = \hat{Q}(\hat{x}, \hat{p}) = \hat{Q}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)$

$\hat{T} = \frac{1}{2}mv^2 = \frac{\hat{p}^2}{2m} = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \cdot \frac{1}{2m} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

$\langle T \rangle = \int \Psi^* \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)\Psi \, dx$

<u>CAREFUL</u> <u>order matters!</u>

back to SE $\quad i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V}\right]\Psi$

?

$\hat{T} + \hat{V} = \hat{H}$

Hamiltonian says $i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$

$$\psi(x,t) = \sqrt{\frac{2}{a}}\,\mathrm{SIN}\left(\frac{\pi x}{a}\right)$$

$\langle P \rangle = ?$ $$\frac{2}{a}\int \mathrm{SIN}\left(\frac{\pi x}{a}\right)\frac{\hbar}{i}\frac{\partial}{\partial x}\mathrm{SIN}(\pi x/a)\,dx$$

$$= \frac{2\hbar}{ai}\int_0^a \mathrm{SIN}\left(\frac{\pi x}{a}\right)\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)dx$$

$$= \frac{2\hbar\pi}{a^2 i}\int_0^a \mathrm{SIN}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)dx$$

Let $u = \mathrm{SIN}\left(\frac{\pi x}{a}\right)$ $du = \frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)dx$

$$= \frac{2\hbar}{ai}\int_0^0 u\,du = 0 \checkmark$$

Or $\langle P \rangle = \dfrac{d\langle x \rangle}{dt} = 0 \checkmark$

$\langle T \rangle ?$ $$\frac{\hat{p}^2}{2m} \rightarrow \frac{2}{a}\int_0^a \mathrm{SIN}(\,)\,\hbar^2\frac{\partial^2}{\partial x^2}\mathrm{SIN}(\,)\,dx \neq 0$$

$$= +\frac{2\hbar^2}{a}\int \mathrm{SIN}(\,)\,\frac{\pi^2}{a^2}\,\mathrm{SIN}(\,)\,dx$$

You Finish