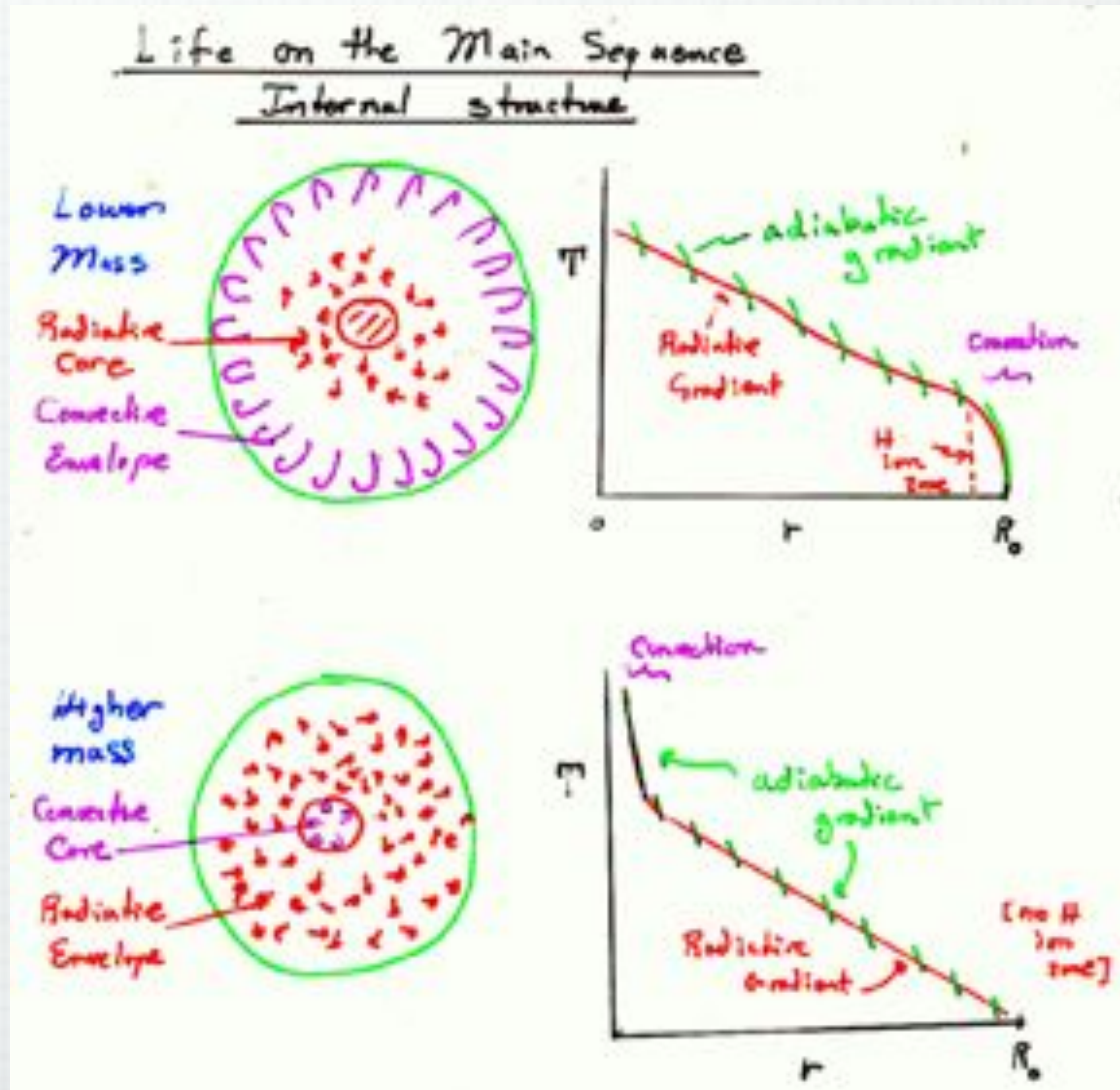
Lecture 6

Stellar Interiors or, how to build a star

Equations of stellar structure

- Hydrostatic equilibrium
- Mass Conservation
- Energy Transport
- Energy Conservation
- Temperature profile depends on mass



We need 4 (and then some) equations

* Coupled ordinary differential equations
* Pressure balance or imbalance
* Mass conservation
* Energy conservation (generation)
* Energy transport, generally cast in terms of temperature

Hydrostatic Equilibrium

* For stars in equilibrium we have seen the equation before

$$\frac{dP(r)}{dr} = \rho(r)[g_{rad}(r) - g(r)] = \rho(r)[g_{rad}(r) - \frac{GM(r)}{r^2}]$$

* Where the radiative acceleration was given in chapter 3

* For variable stars the star is out of hydrostatic equilibrium and we must defer to Newton's law

$$\rho\frac{d^2r}{dt^2} = \rho(r)[g_{rad}(r) - \frac{GM(r)}{r^2}] - \frac{dP(r)}{dr}$$

Mass conservation

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

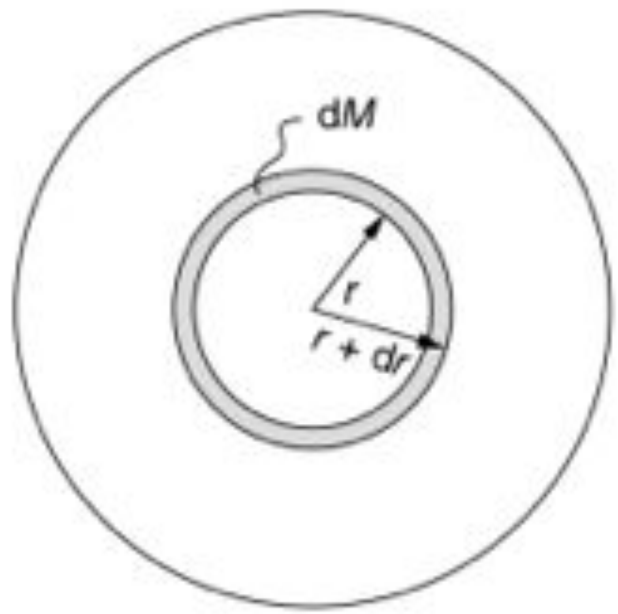


Figure 5.1 Illustration of the mass dM found inside the spherical shell found between the radii r and r + dr (shaded area).

Where the integral from 0 to R must give the mass of the star

Energy transport

* Three modes exist
  * Convection, conduction, radiation
  * Conduction will be largely ignored
  * We will first look at purely radiative transport and add convection later

* The Eddington flux is given by

$$H(r) = -\frac{1}{3k_r\rho}\frac{dB}{dT}\frac{dT}{dR} = \frac{1}{4\pi}\frac{L(r)}{4\pi r^2}$$

* Which depends on r as the energy generation rate depends on the location within the star.

$$B(r) = \frac{\sigma T(r)^4}{\pi}$$

* Leading to $$\frac{dT(r)}{dr} = -\frac{3k_r\rho}{64\pi r^2\sigma T^3}L(r)$$
* Which says the luminosity is proportional to the temperature gradient and that a temperature gradient is necessary for energy transport and that the gradient is proportional to opacity
* Be careful, this is radiative energy transport only, more on other modes later

Radiative temperature gradient

$$\frac{dln(T(r))}{dln(P(r))} = \frac{3k_r P(r)}{64\pi r^2 g\sigma T^4} L(r) = \nabla_{rad}$$

* we will later see that the steepness of the temperature gradient leads to either radiation or convection dominating energy transport

Energy conservation

* The energy generation at a given radius depends on the thermonuclear reactions occurring at a given radius and the amount of material participating in the generation of energy

* $\varepsilon(r)\rho(r)$ is the energy generated per unit time per unit volume at r. We will derive this explicitly in chapter 6

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \epsilon(r)$$

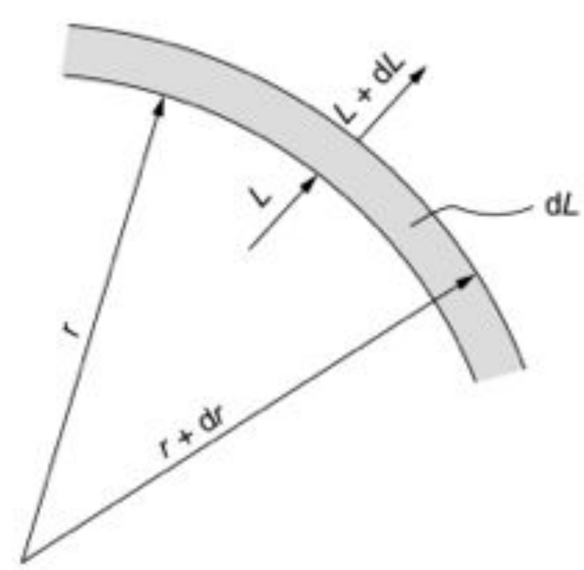


Figure 5.2 Illustration of the energy generated per unit time dL inside the spherical shell found between the radii r and r + dr (shaded area).

Energy generation

* Primarily occurs in the core due to fusion
* Dies of at about 25% of the stellar radius

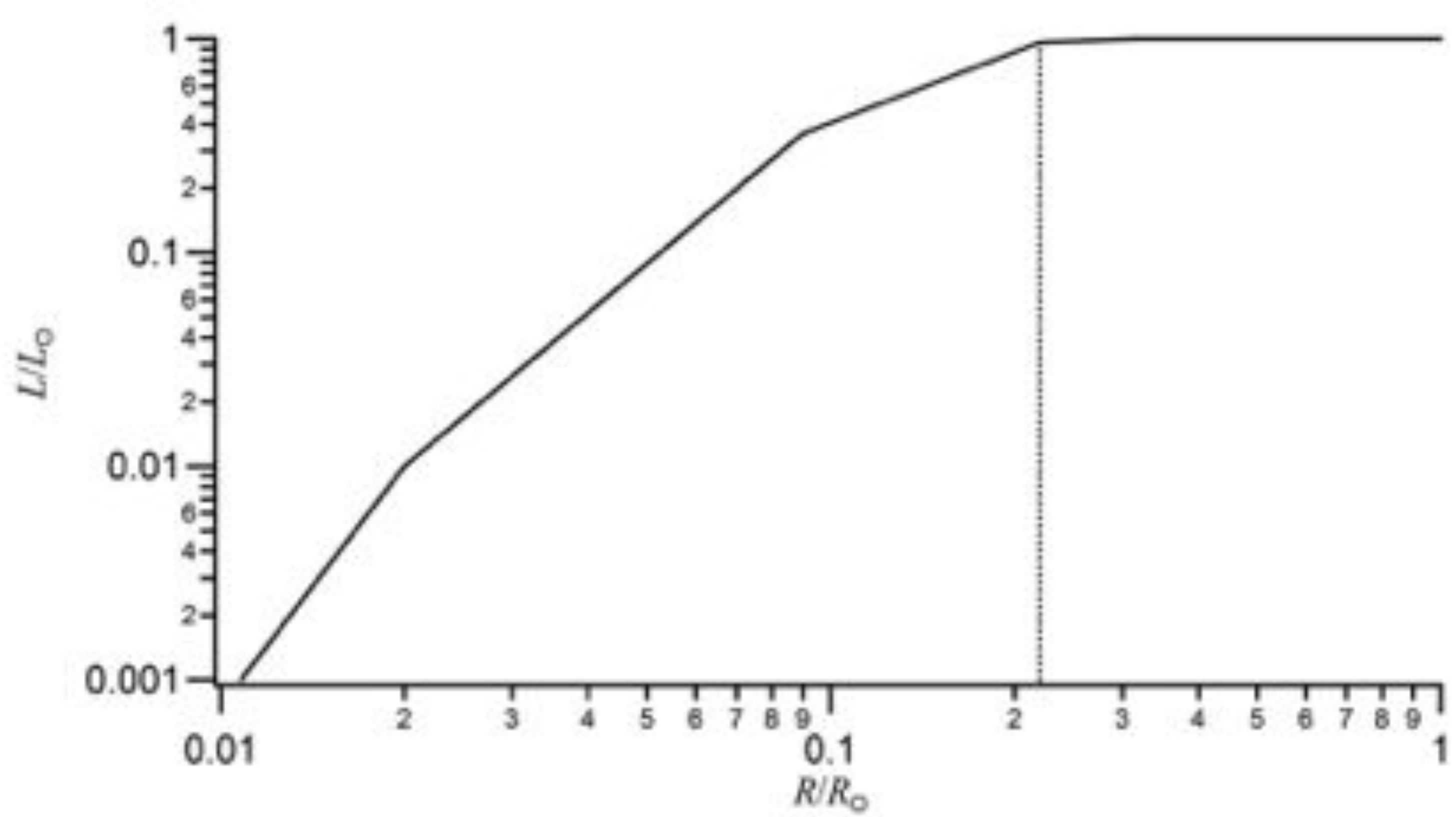


Figure 5.3 Luminosity as a function of radius inside the Sun. The data used here are those found in Table 5.1 (see Section 5.5). The dotted line shows approximately the radius where $\varepsilon \to 0$ or $L \to L_\odot$.

Final Ingredients

* Must specify pressure, opacity, and energy generation as equations depending on density, temperature, and mass fraction of elements (ρ,T,X_i)
* Must specify boundary conditions
    * L(r=0)=0, M(r=0)=0, Pressure at surface = 0, Density at surface = 0, temperature at surface = T_{eff}

In Practice

* We change all of these equations to be dependent on mass enclosed so we can write them as functions of optical depth

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4(M)}$$

$$\frac{dr(M)}{dM} = \frac{1}{4\pi r^2 \rho(M)}$$

$$\frac{dL}{dM} = \epsilon(M)$$

$$\frac{dT(M)}{dM} = -\frac{3k_R}{256\pi^2 r^4 \sigma T^3} L(M)$$

How to build a star

* Construct a star with a given metallicity
* Solve the coupled system of equations for hydrostatic equilibrium
* Update the metallicity which in turn changes the energy generation rate
* Iterate
* http://www.astro.wisc.edu/~townsend/static.php?ref=ez-web
* http://www.astro.uni-bonn.de/~izzard/window.html
* http://mesa.sourceforge.net/index.html

In reality

* A 4th energy transport mode exists, neutrino losses.
* Neutrinos are generated in nuclear fusion
* They have an exceedingly low optical depth, 1 scattering per parsec of stella material
* This results in a loss term in the energy generation rate

Monochromatic flux in stellar interiors

* Global elements depend only on integrated flux, radiative diffusion of mass fraction of elements depends on local flux

* Equating the local luminosity to be

$$L(r) = 4\pi R^2 \sigma T_{eff}^4$$

* Let's us rewrite dT/dr as

$$\frac{dT}{dr} = -\frac{3k_r \rho R^2 T_{eff}^4}{16 r^2 T^3}$$

* Using this result in the equation for the Eddington flux valid at large optical depths results is

$$H_\nu = \frac{1}{16}\frac{k_r}{k_\nu}\frac{T_{eff}^4}{T^3}(\frac{R}{r})^2\frac{dB_\nu}{dT}$$

* Using (see hw 5.5) $$\frac{dB_\nu}{dT} = \frac{2k_B^3T^2}{h^2c^2}[\frac{u^4e^u}{(e^u-1)^2}]$$
* with $u = \frac{h\nu}{k_BT}$
* Results in $$H_\nu = \frac{k_B^3}{8h^2c^2}\frac{k_r}{k_\nu}\frac{T_{eff}^4}{T^3}(\frac{R}{r})^2P(u)$$
* Where $$P(u) = [\frac{u^4e^u}{(e^u-1)^2}]$$

* Now the monochromatic flux depends on the effective temperature, the radius within the star, and the local temperature as well as the monochromatic opacity

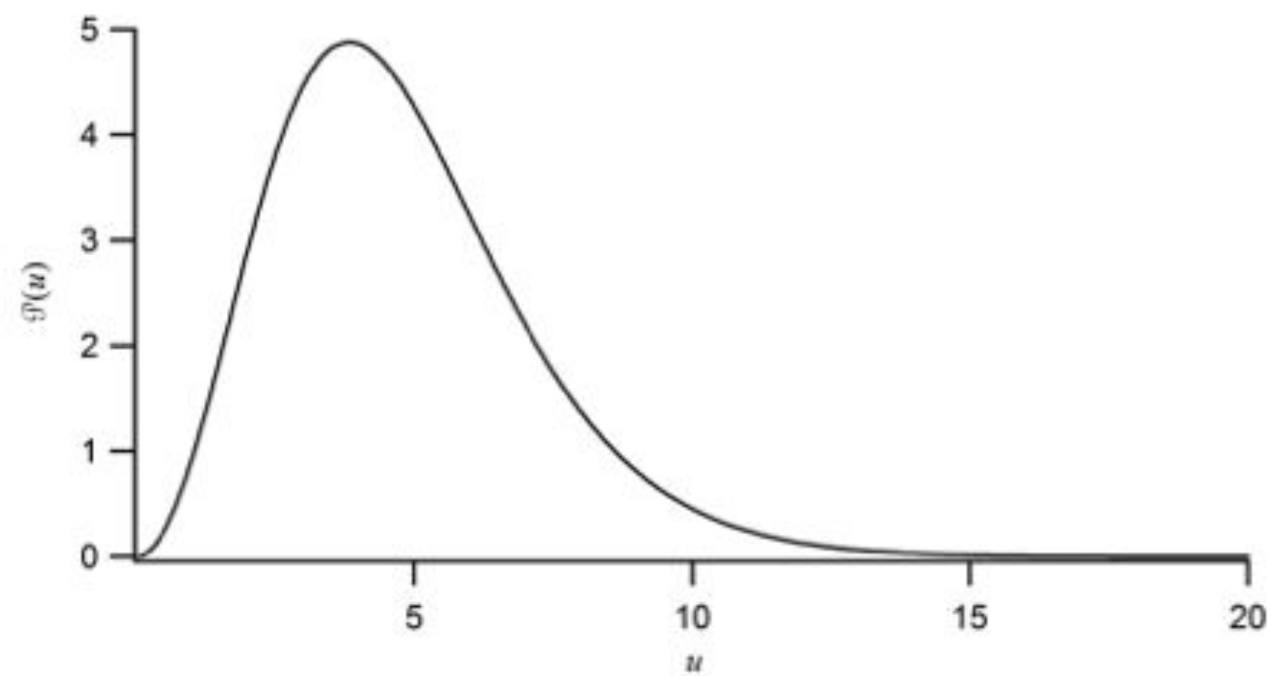


Figure 5.4 The dependence of $\mathcal{P}(u)$ as a function of u. Its maximum is found at $u \approx 3.8$.

?

* P(u) peaks at u=3.83, numerical problem in the homework
* It tells you what frequency or wavelength dominates the flux of radiation transport at a given temperature or, since they are related, a given radius in the star
* For $T=10^7$ K (center of the star) the peak wavelength for P(u) is 3.8 angstroms (x-ray photons)
* For $T=10^4$ K (near the surface) the peak wavelength is 3800 angstroms or in the optical
* We see visible photons emanating from the photosphere, this is why

Energy transport mechanisms

* Conduction, convection, radiation
    * Transport by particle collisions
    * Transport by fluid overturning
    * Transport by radiation
* The temperature gradient (as well as other effects) locally sets which of these is most important for energy transport.
* This is generally radiative or convective

Conduction

* Conduction is important when the mean free path of particle collisions is of the order of the scale on which the temperature changes
* This is generally not important in stars but IS important in degenerate matter remnants such as white dwarfs

How to determine importance?

* We define a modified Eddington flux due to both the radiative flux and conductive flux of energy using a modified opacity defined by

$$\frac{1}{k_{tot}} = \frac{1}{k_{cond}} + \frac{1}{k_{rad}}$$

* This leads to

$$H_{tot} = -\frac{4\sigma T^3}{3\pi k_{tot}\rho}\frac{dT}{dr}$$

* Energy will be preferentially transferred by the pathway with the lower all overall opacity
* This is invariably radiative transport in regular stars

So, convection or radiation

* In smaller stars the temperature gradient is not too steep and radiation dominates the transport of energy until near the surface where convection becomes the dominant mechanism
* In larger stars the situation reverses and stars transport energy convectively near the core and radiatively farther out
* First, some preliminaries

Preliminaries

* Some thermodynamics
* Conservation of energy for a blob
* dU=dQ-dW
* The change of the internal energy of a parcel of gas is given by the change in heat Q added or subtracted from it minus the work the parcel does on its surroundings. If the parcel expands it does positive work and looses energy, if it contracts is has work done on it and gains energy.
* If heat is added internal energy increases, if heat is removed it decreases
* A rising parcel of air expands, does work, and looses energy hence its temperature goes down.
    * It gets colder by about 3-6 degrees F for every thousand feet you go up

* For a monatomic gas the internal energy U is given by the average energy per particle times the number of particles per mass or average energy per average mass $U = \frac{\bar{K}}{\bar{m}}$ $\bar{m} = \mu m_H$
* For an ideal gas $\bar{K} = \frac{3k_BT}{2}$
* So $U = \frac{3}{2}\frac{k_B}{\mu m_H}T = \frac{3}{2}nRT$

Specific Heats

* The change in the internal heat Q of mass element m is given in terms of the specific heats C

$$C_p = \frac{\partial Q}{\partial T}|_p \quad C_v = \frac{\partial Q}{\partial T}|_v$$

* Where these denote the amount of heat necessary to raise the unit temperature of a unit mass of material at either a fixed pressure of fixed volume
* For example, it requires about 1000 joules of energy to raise the temperature of one kilogram of air by one degree and 4186 joules to raise the temperature of one kilogram of water by one degree
* Ever wonder why humid climates feel hotter? Water holds heat better

* Consider a blob of gas to exist in a cylinder with cross sectional area A doing a differential amount of work on it's surroundings given by dW
* If the cylinder is filled with gas of mass m and pressure P the gas exerts a force F=PA as it moves the cylinder upwards through dr
* The work per unit mass may then be defined as

$$dW = \frac{F}{m}dr = \frac{PA}{m}dr = PdV$$

* This is positive work if dV is positive and negative if dV is negative so a contraction increases the internal energy and an expansion decreases it

- The law of conservation of energy may therefore be recast as dU=dQ-PdV
- At a constant volume

$$dU = \frac{\partial Q}{\partial T}|_V dT = C_V dT$$

- Therefore $$dU = \frac{3nR}{2} dT = C_V dT$$
- At a constant pressure

$$dU = \frac{\partial Q}{\partial T}|_P dT - P\frac{\partial V}{\partial T}|_P dT$$

* But PV=nRT so d(PV)=PdV+VdP=RT(dn)+nr(dT)
* For constant P and n (constant pressure and number density) PdV=nr(dT)
* or $P\frac{\partial V}{\partial T}|_P = nR$
* and, using $dU = \frac{\partial Q}{\partial T}|_V dT = C_V dT$ and $dU = \frac{\partial Q}{\partial T}|_P dT - P\frac{\partial V}{\partial T}|_P dT$
* gives $C_P = C_V + nR$
* This holds only for an ideal gas
* Define $\gamma = 1 + \frac{1}{n} = 1 + \frac{2}{f} = \frac{C_P}{C_v}$
* where this n is the polytropic index an not the number density and f is the degrees of freedom, 3 for a monatomic gas and 5 for a diatomic gas (3D plus 2D rotation)

Adiabatic Gas Law

* We will consider the case where we have an adiabatic process $dQ = 0$, no heat is exchanged between a parcel of gas and it's surroundings
* Then $dU=-PdV$
* Recalling $dU=C_v dT$ we have $dT=-PdV/C_v$
* With constant n we have $PdV+VdP=nRdT=-nRPdV/C_v$
* Or $-PdV(1+nR/C_v)=VdP$
* But $\gamma = 1 + \frac{1}{n} = 1 + \frac{2}{f} = \frac{C_P}{C_v}$
* and $C_P = C_V + nR$
* So $\gamma = 1 + \frac{nR}{C_V}$

Adiabatic Gas Law

* Leading to $-PdV\gamma = VdP \rightarrow \gamma\frac{dV}{V} = -\frac{dP}{P}$
* Which may be integrated to yield $PV^{\gamma} = K$
* Or, on using the ideal gas law PV=nRT $P = K'T^{\frac{\gamma}{\gamma-1}}$
* The adiabatic sound speed is defined as

$$\sqrt{\frac{\gamma P}{\rho}}$$

* For the sun, an average sound speed may be calculated from an average pressure and density to be 4×10^{5} m s^{-1} which leads to a sound wave crossing time of about 29 minutes

Adiabatic Temperature Gradient

* Recalling the ideal gas law $P = \frac{\rho K_B T}{\mu m_H}$ we can derive an expression for the adiabatic temperature gradient

$$\frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

* Recalling $V = \frac{1}{\rho}$ is the specific volume we may rewrite the adiabatic gas law as $P = K\rho^\gamma$
* This gives $\frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr}$
* Assuming the mean molecular weight is constant kills the first term in the pressure gradient and gives

$$\frac{dT}{dr}|_{adiabatic} = (1 - \frac{1}{\gamma})\frac{T}{P}\frac{dP}{dr} = -(1 - \frac{1}{\gamma})\frac{\mu m_h}{k_B}\frac{GM_r}{r^2}$$

* Where we have used one of the equations of stellar structure and neglected radiative pressure

Adiabatic Temperature Gradient

* With a bit of math and some previously derived identities this may be recast into
$$\frac{dT}{dr}\Big|_{adiabatic} = -\frac{g(r)}{C_P}$$

* This equation describes how the temperature changes as the bubble rises and expands adiabatically. Remember, adiabatic expansion implies no heat transfer with the surroundings but as the bubble expands the net temperature decreases to maintain the overall internal energy

* In a star, whenever the temperature gradient is steeper than the adiabatic temperature gradient, energy will be transported adiabatically through convective uplifting of parcels of plasma

Convection

and the Schwarzschild criterion

* Cells rise from hotter to cooler regions of stars without releasing energy (adiabatically), this is essentially Archimedes principle, bouyancy

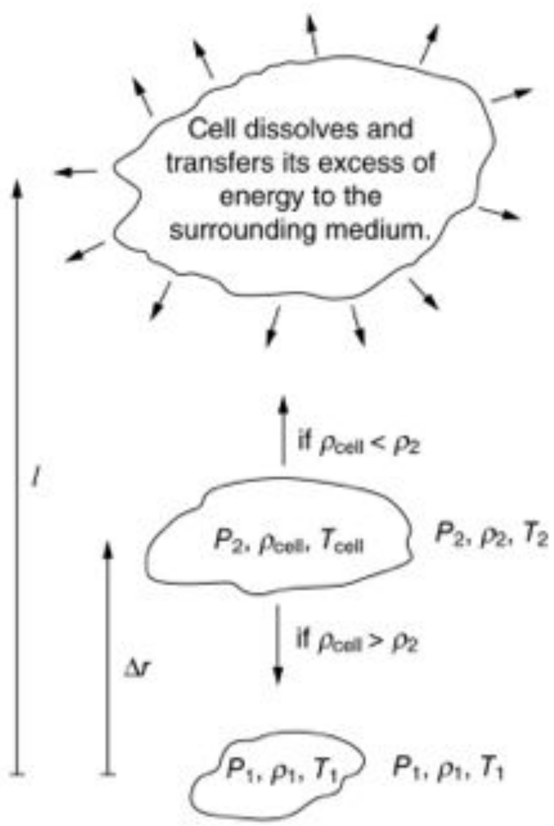
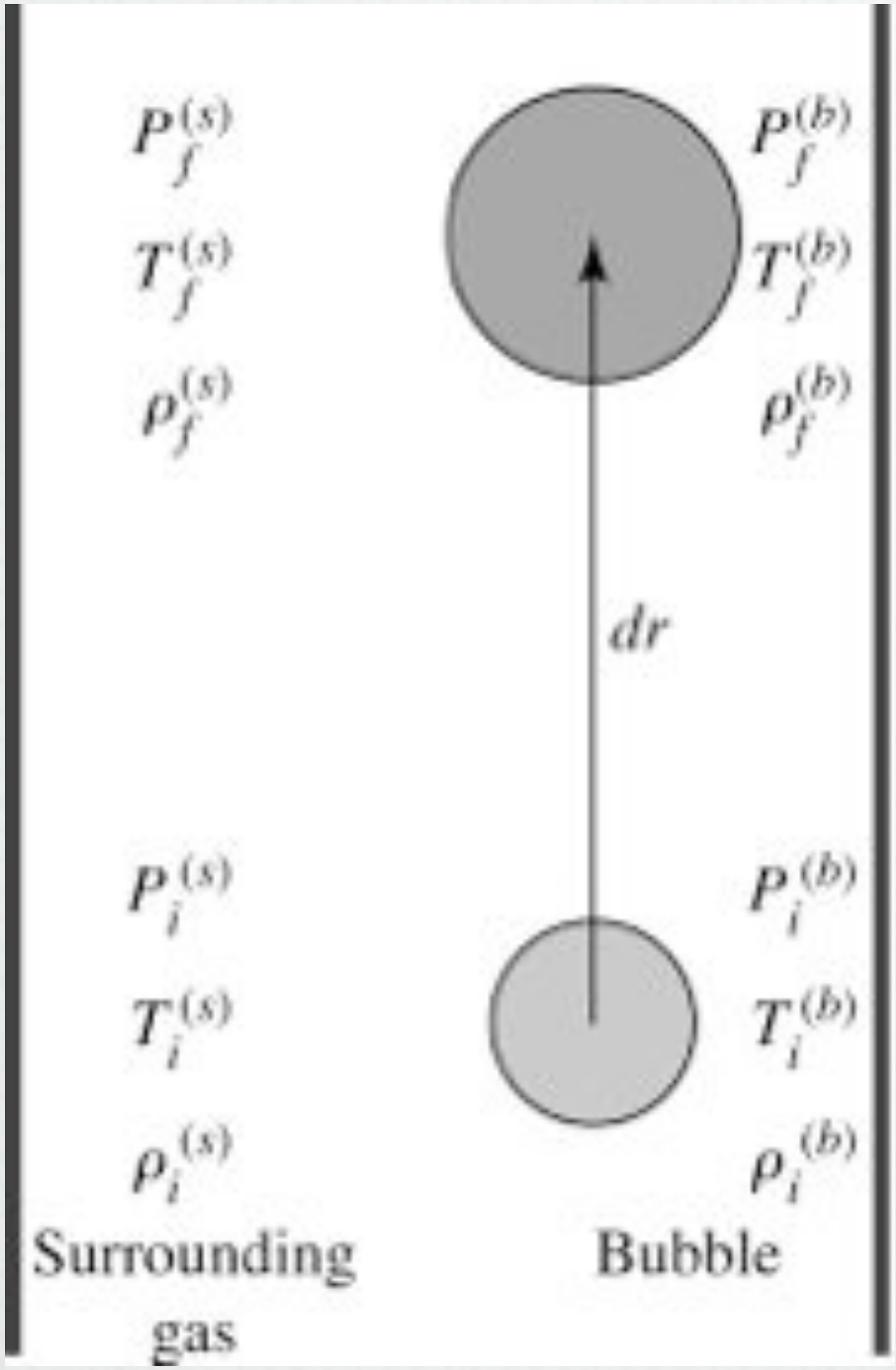


Figure 5.5 Illustration of the convection process. A convective cell found in the bottom part of this figure is displaced by a distance Δr. Initially the values of the pressure, density and temperature inside the cell are equal to the corresponding values found in the surrounding medium. During its displacement the pressure inside the cell is always equal to the pressure in the medium, however, its density and temperature change. After the displacement, if the density in the cell is larger than the density of the medium the cell sinks back toward its original position. If the density in the cell is smaller than the density in the surrounding medium, the cell is buoyant and travels a certain distance l (called the mixing length) before dissolving. It then transfers its excess of energy to the surrounding medium.

Figure 10.10 of Introduction to Modern Astrophysics, Pearson 2009



b refers to the bubble's quantities
s to the surrounding gas

The criteria for a parcel of gas displaced an infinitesimal distance dr from equilibrium is that that the bubble's final density be less than that of the surrounding medium

If this condition is met then there will be a net force acting to continuously displace the bubble upwards until it reaches a region where its density is in equilibrium with the surrounding gas

- The net force on a parcel of gas due to its own weight and buoyancy is given $f_{net} = -g\delta\rho$ where $\delta\rho = \rho_i^b - \rho_i^s < 0$
- We assume a slow adiabatic expansion which requires the initial temperature and density of the bubble and its surroundings to be equal and the bubble rises slowly so that the final pressure of the surroundings and the bubble are in equilibrium.
- We may express the final density of the bubble and surrounding gas as a Taylor series expansion

$$\rho_f^b \approx \rho_i^b + \frac{d\rho}{dr}|^b dr \qquad \rho_f^s \approx \rho_i^s + \frac{d\rho}{dr}|^s dr$$

- Now if $\rho_f^b < \rho_f^s$ convection occurs giving the criteria for convection to be $$\frac{d\rho}{dr}|^b < \frac{d\rho}{dr}|^s$$

* Recalling $\frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$
* And assuming the mean molecular weight is constant we rewrite the criteria for convection in terms of the surrounding gas's quantities

$$\frac{1}{\gamma}\frac{\rho_i^b}{P_i^b}\frac{dP}{dr}|^b < \frac{\rho_i^s}{P_i^s}[\frac{dP}{dr}|^s - \frac{P_i^s}{T_i^s}\frac{dT}{dr}|^s]$$

* Recalling the bubble and surrounding gas are in pressure balance and the initial surrounding and bubble temperature are equivalent we arrive at

$$(\frac{1}{\gamma} - 1)\frac{dP}{dr} < -\frac{P}{T}\frac{dT}{dr}|_{actual}$$

* or

$$(\frac{1}{\gamma} - 1)\frac{T}{P}\frac{dP}{dr} > \frac{dT}{dr}|_{actual}$$

* but $\frac{dT}{dr}|_{adiabatic} = (1 - \frac{1}{\gamma})\frac{T}{P}\frac{dP}{dr} = -(1 - \frac{1}{\gamma})\frac{\mu m_h}{k_B}\frac{GM_r}{r^2}$
* So the requirement for convection is that

$$\frac{dT}{dr}|_{adiabatic} > \frac{dT}{dr}|_{actual}$$

* Which simply means that when the gradient is steep enough convective transport will ensue.
* In reality the temperature gradient is always negative, stellar temperatures decrease outwardly from the core

- Taking absolute values $|\frac{dT}{dr}|_{adiabatic}| < |\frac{dT}{dr}|_{actual}|$
- Leads to $\frac{dlnP}{dlnT} < \frac{\gamma}{\gamma - 1}$
- For a monatomic gas gamma is 5/3 and the quantity on the right is 5/2 = 2.5

- In general three specific things cause convection to occur
  - A large specific heat $\frac{dT}{dr}|_{adiabatic} = -\frac{g(r)}{C_P}$
  - High opacity or energy flux

$$\frac{dlnP}{dlnT}_{rad} = \nabla_{rad} = \frac{3k_r}{64\pi r^2 g}\frac{P}{\sigma T^4}L(r) > \frac{\gamma}{\gamma - 1}$$

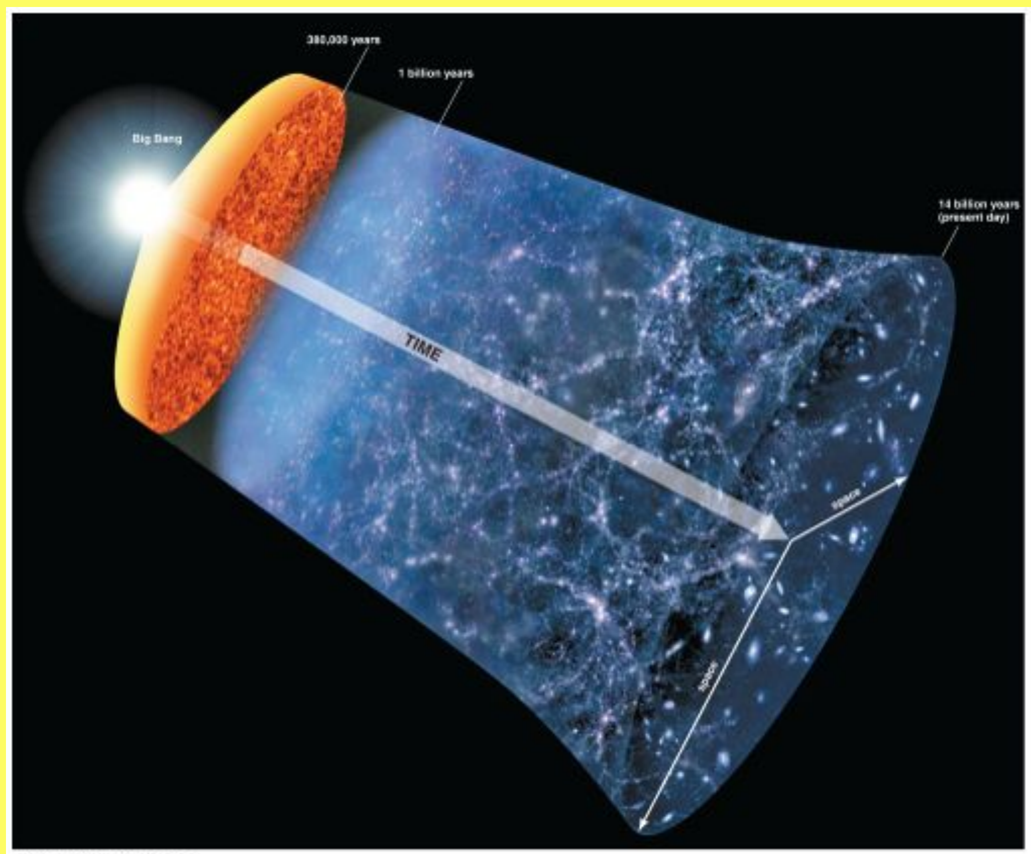
  - Steep dependence on temperature of the energy generation rate such as in CNO fusing stars or stars past the main sequence undergoing triple alpha burning which

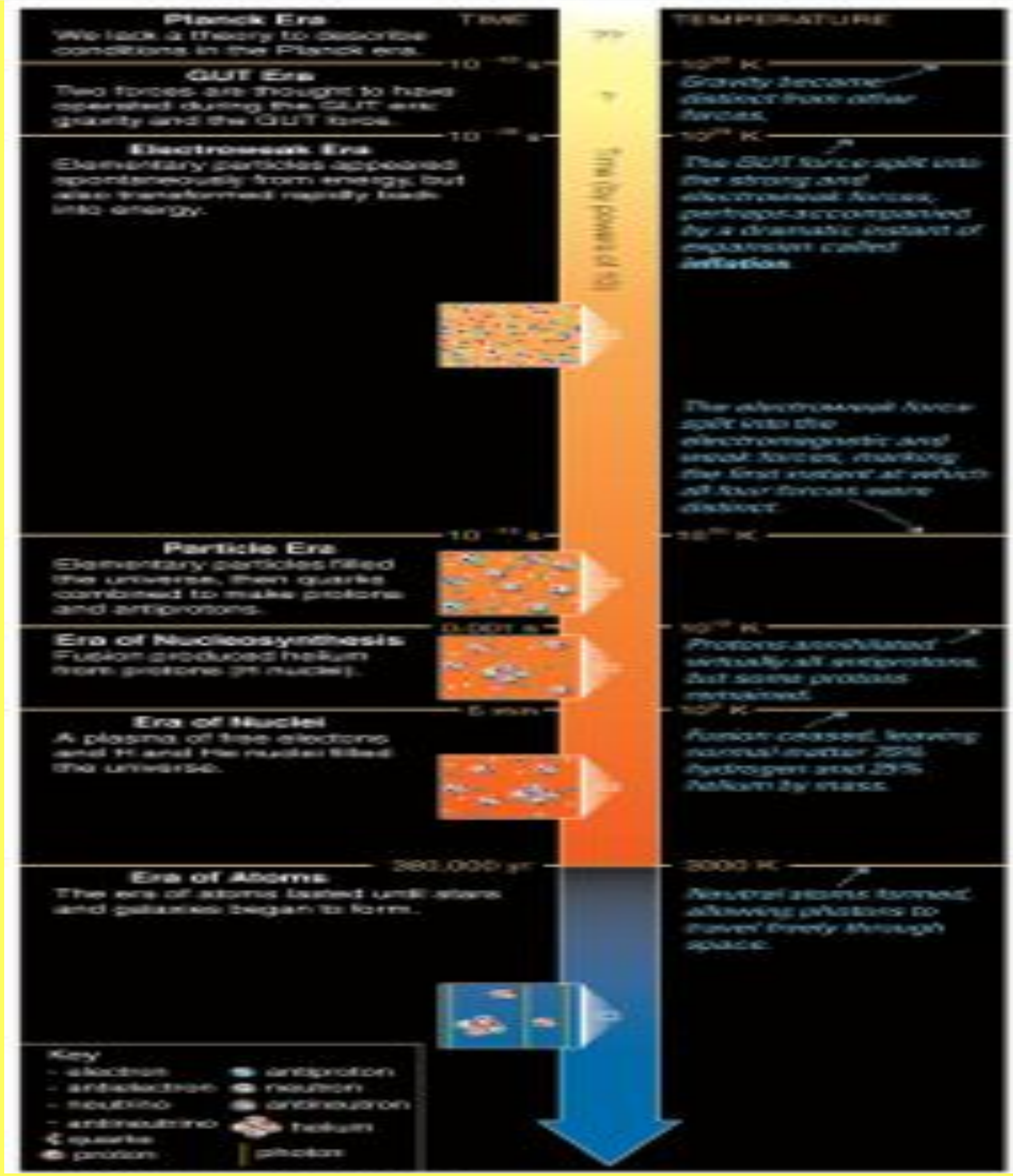
Segue

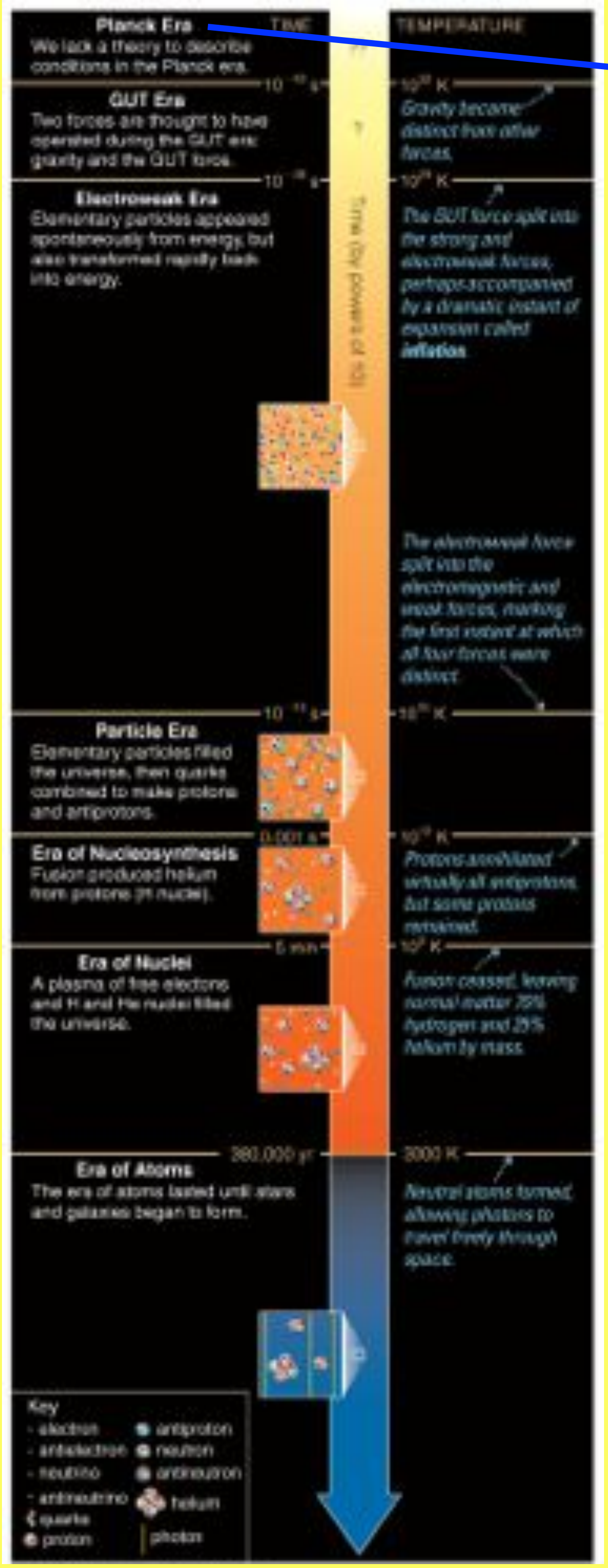
* Reionization and the first stars
* If the universe started with no metals how do we get planets?

We'll start with right after the beginning

What is the history of the universe according to the Big Bang theory?



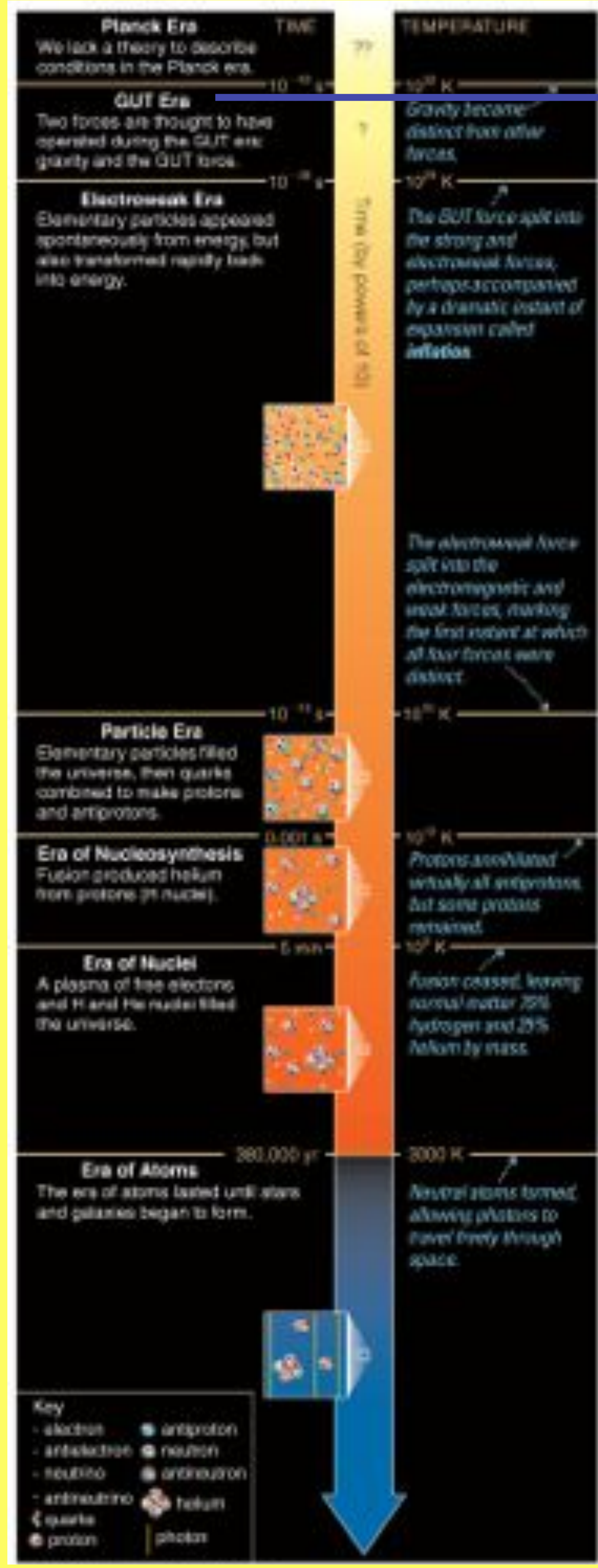

Planck Era
TIME
TEMPERATURE
GUT Era
Electroweak Era
Particle Era
Era of Nucleosynthesis
Era of Nuclei
Era of Atoms
Key



Planck era

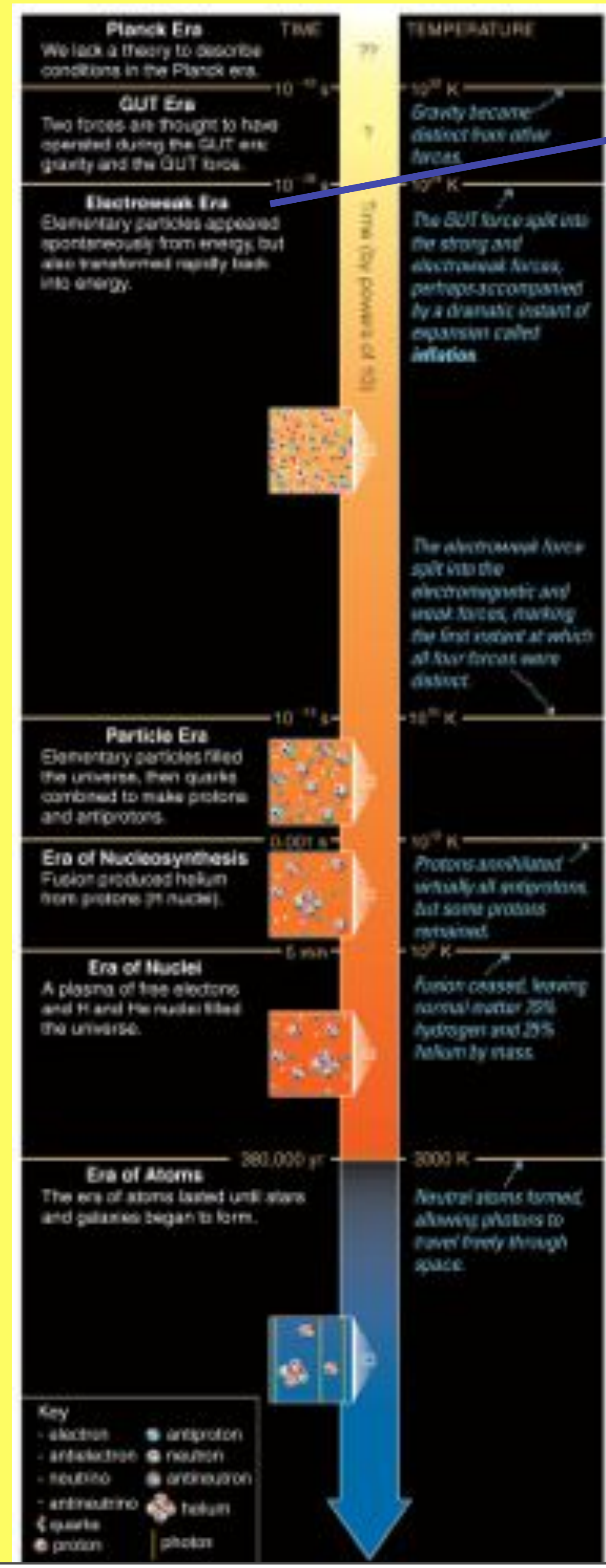
Before Planck time ($\sim 10^{-43}$ second)

No theory of quantum gravity



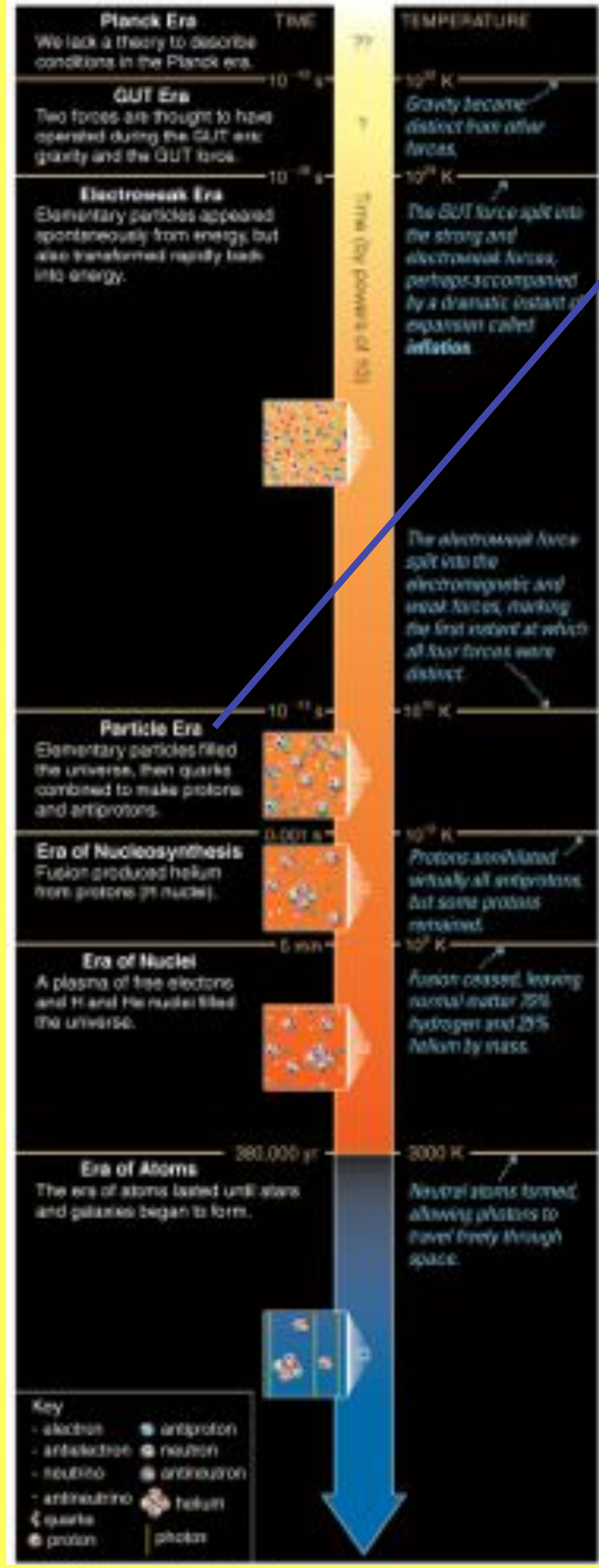
GUT era

Lasts from Planck time ($\sim 10^{-43}$ second) to end of GUT force ($\sim 10^{-38}$ second)



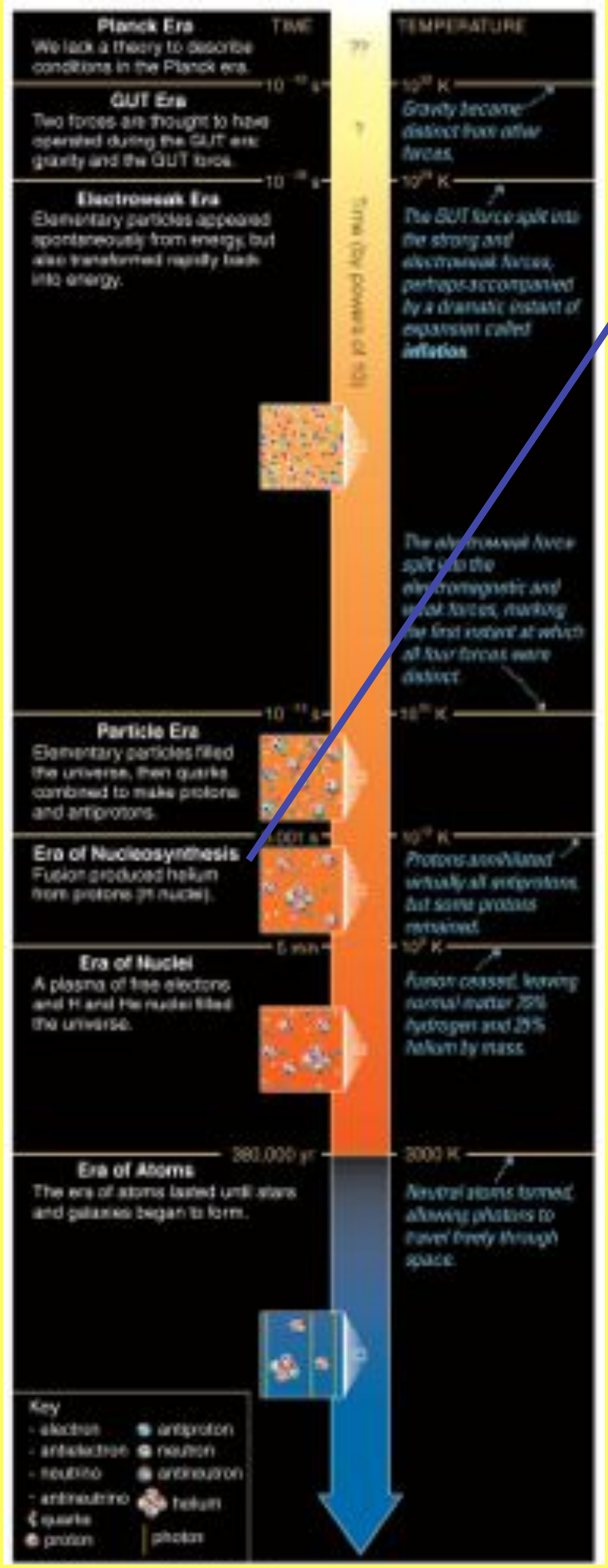
Electroweak era

Lasts from end of GUT force (~10^{-38} second) to end of electroweak force (~10^{-10} second).



Particle era

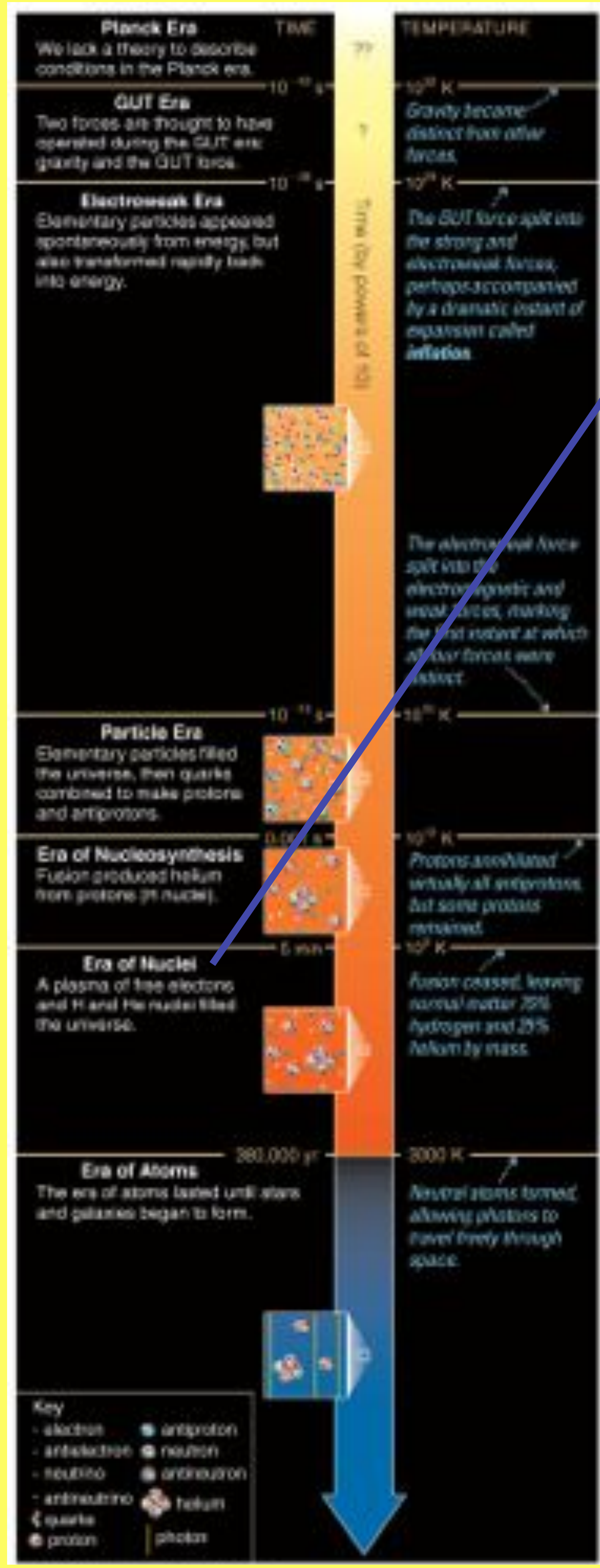
Amounts of matter and antimatter nearly equal (roughly 1 extra proton for every 10^9 proton–antiproton pairs!)



Era of nucleosynthesis

Begins when matter annihilates remaining antimatter at ~ 0.001 second.

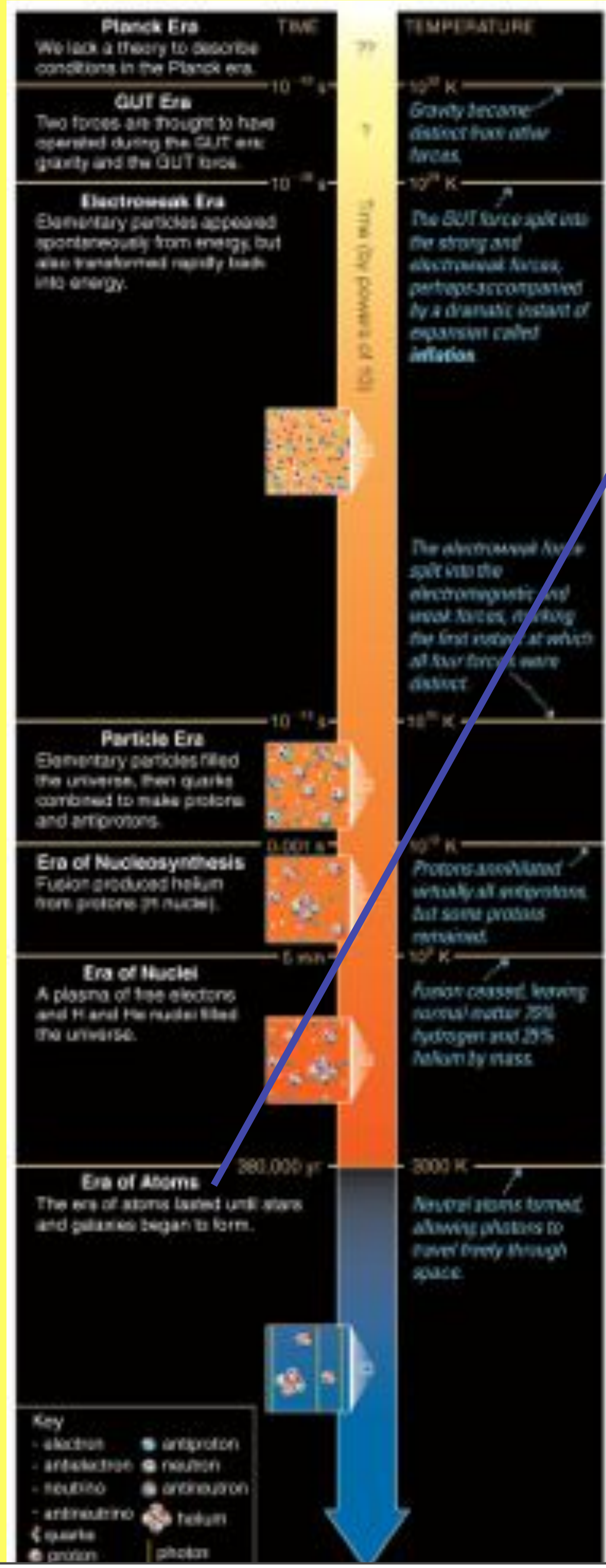
Nuclei begin to fuse.



Era of nuclei

Helium nuclei form at age ~ 3 minutes.

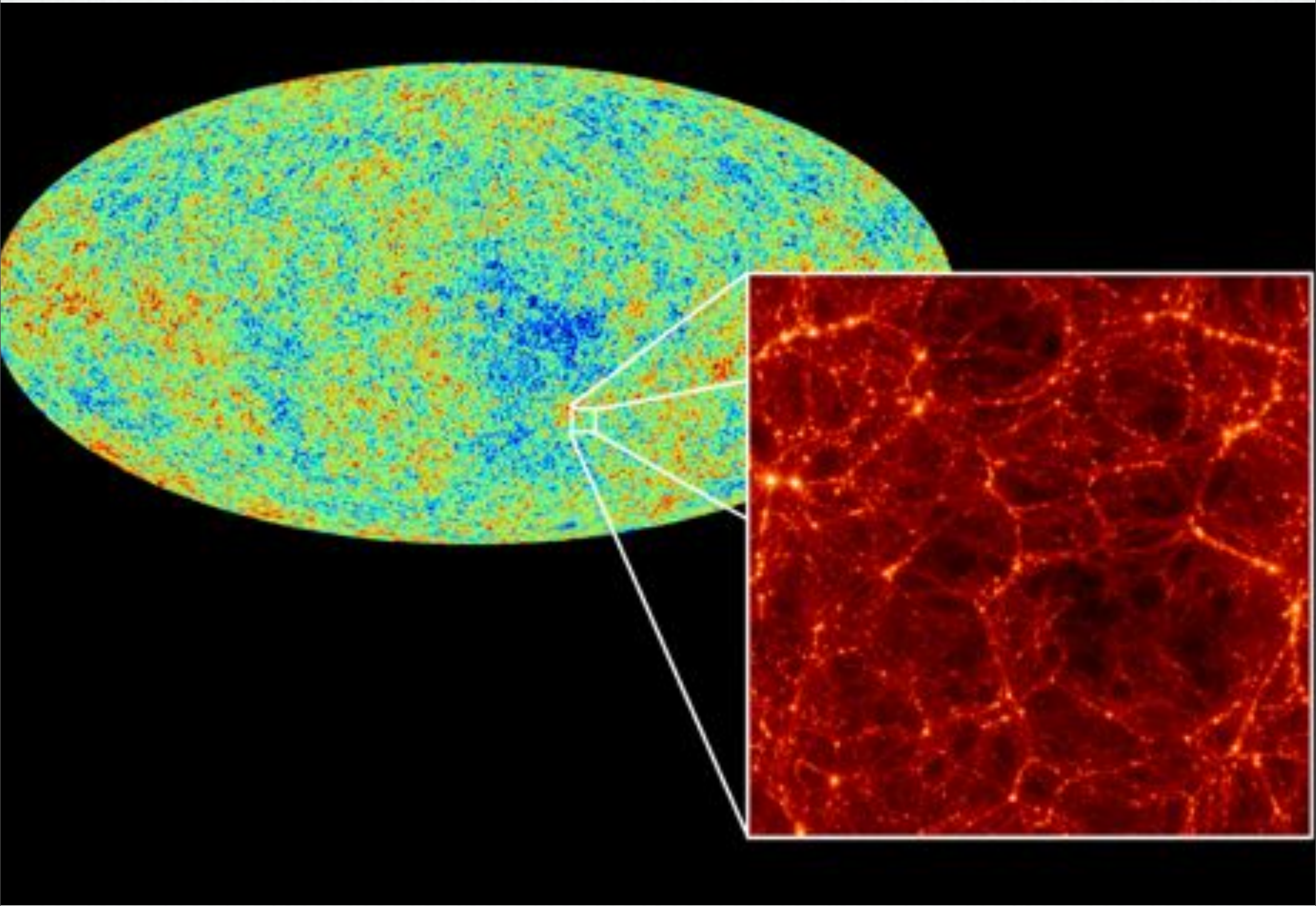
Universe became too cool to blast helium apart.

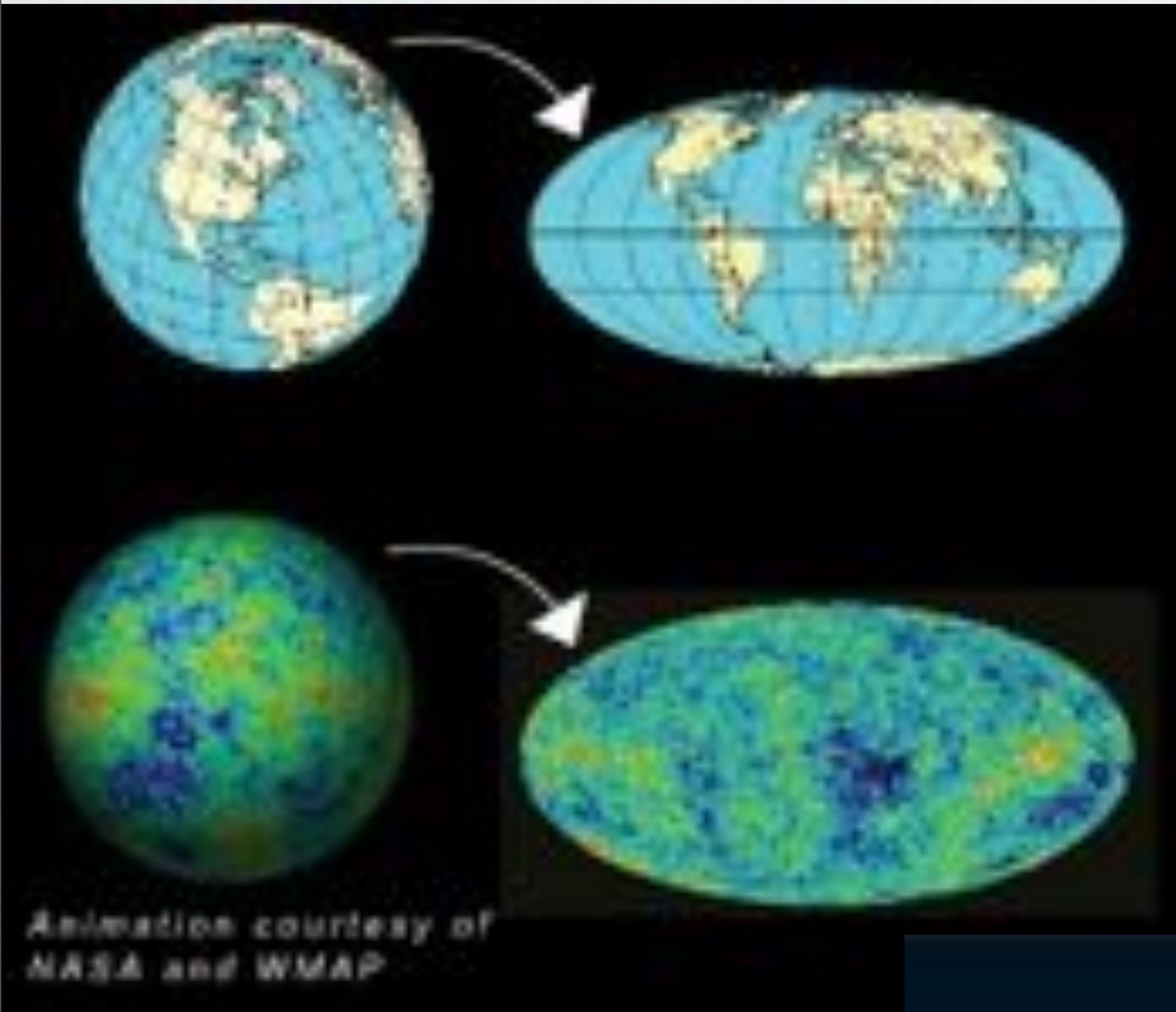


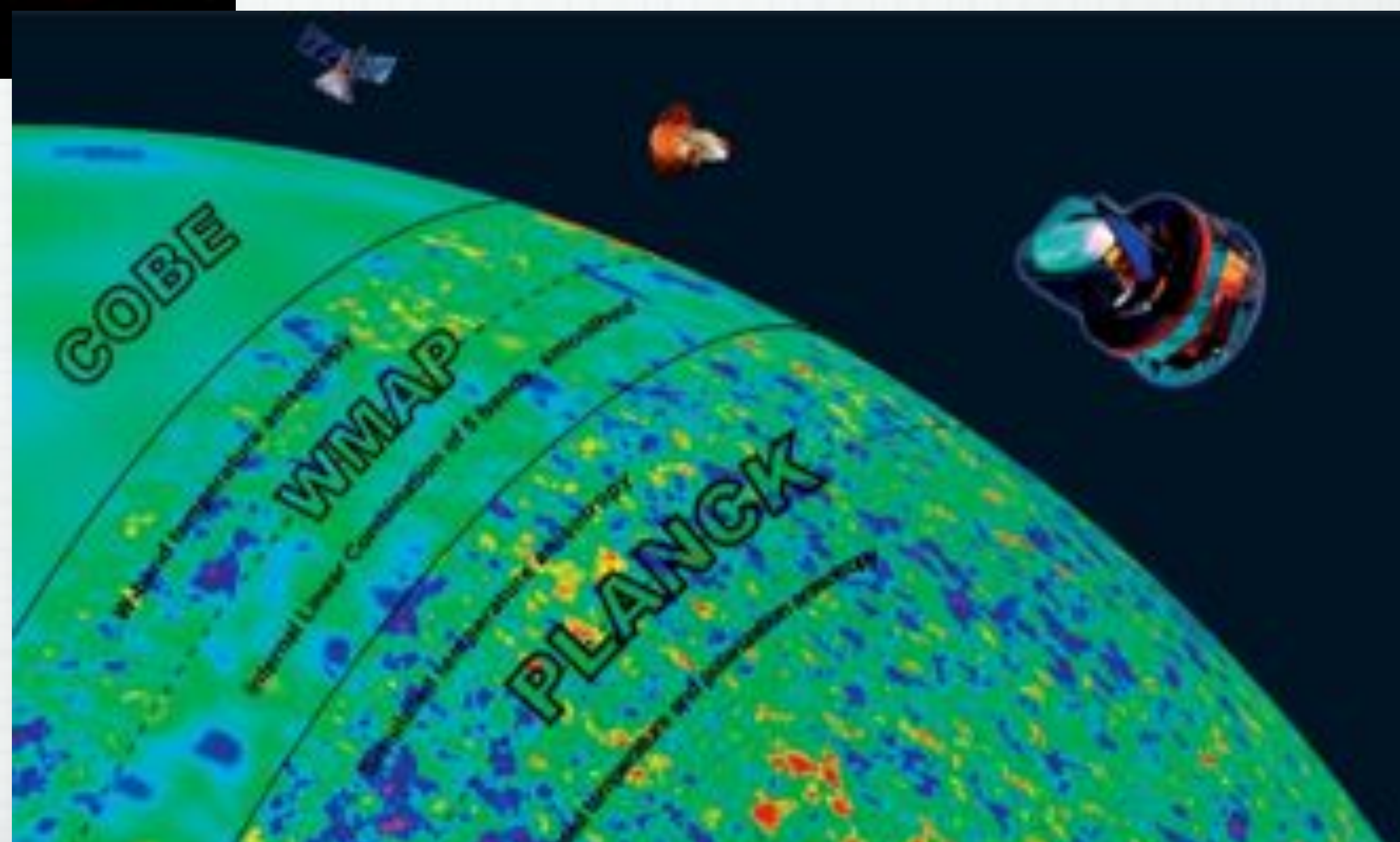
Era of atoms

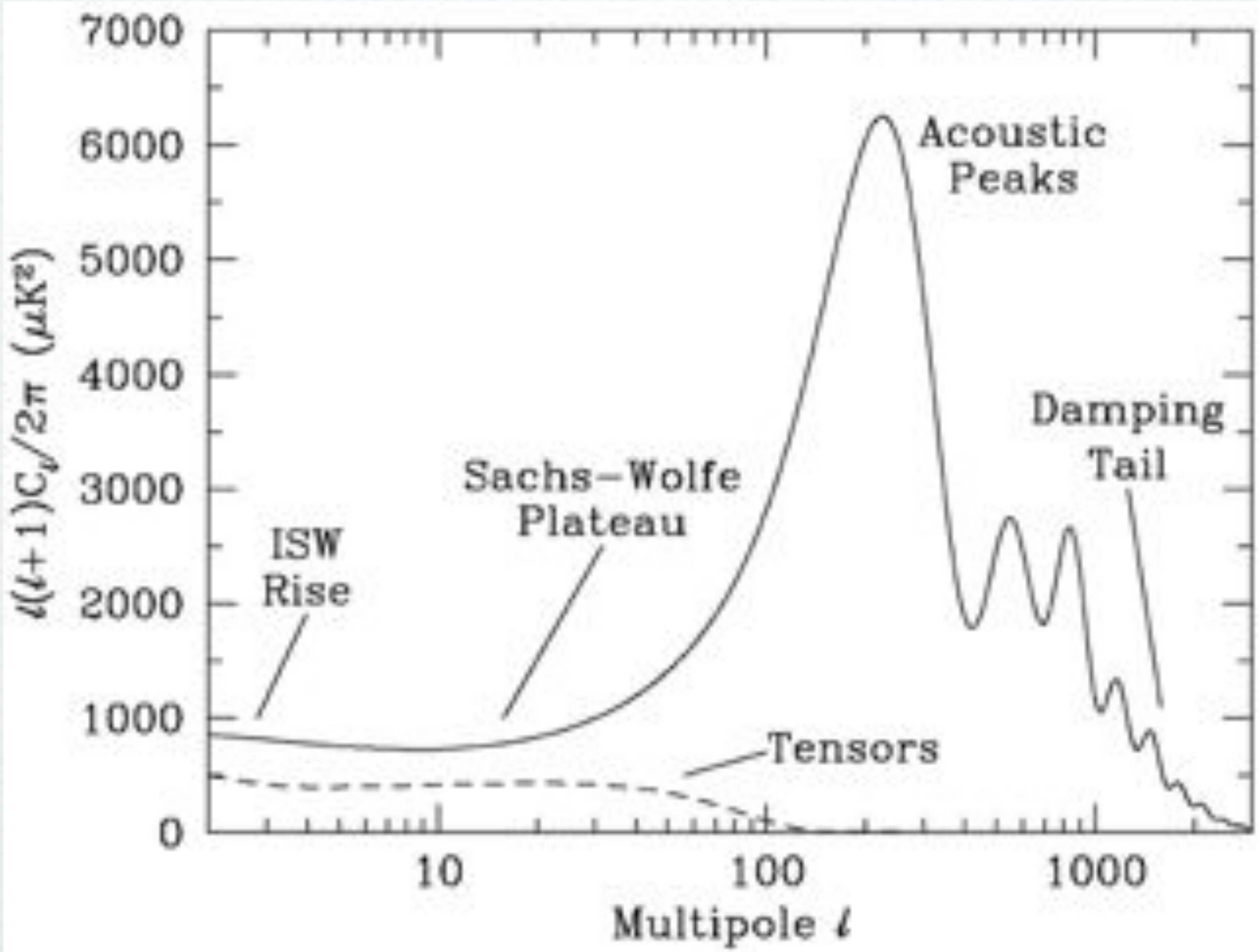
Atoms form at age ~ 380,000 years.

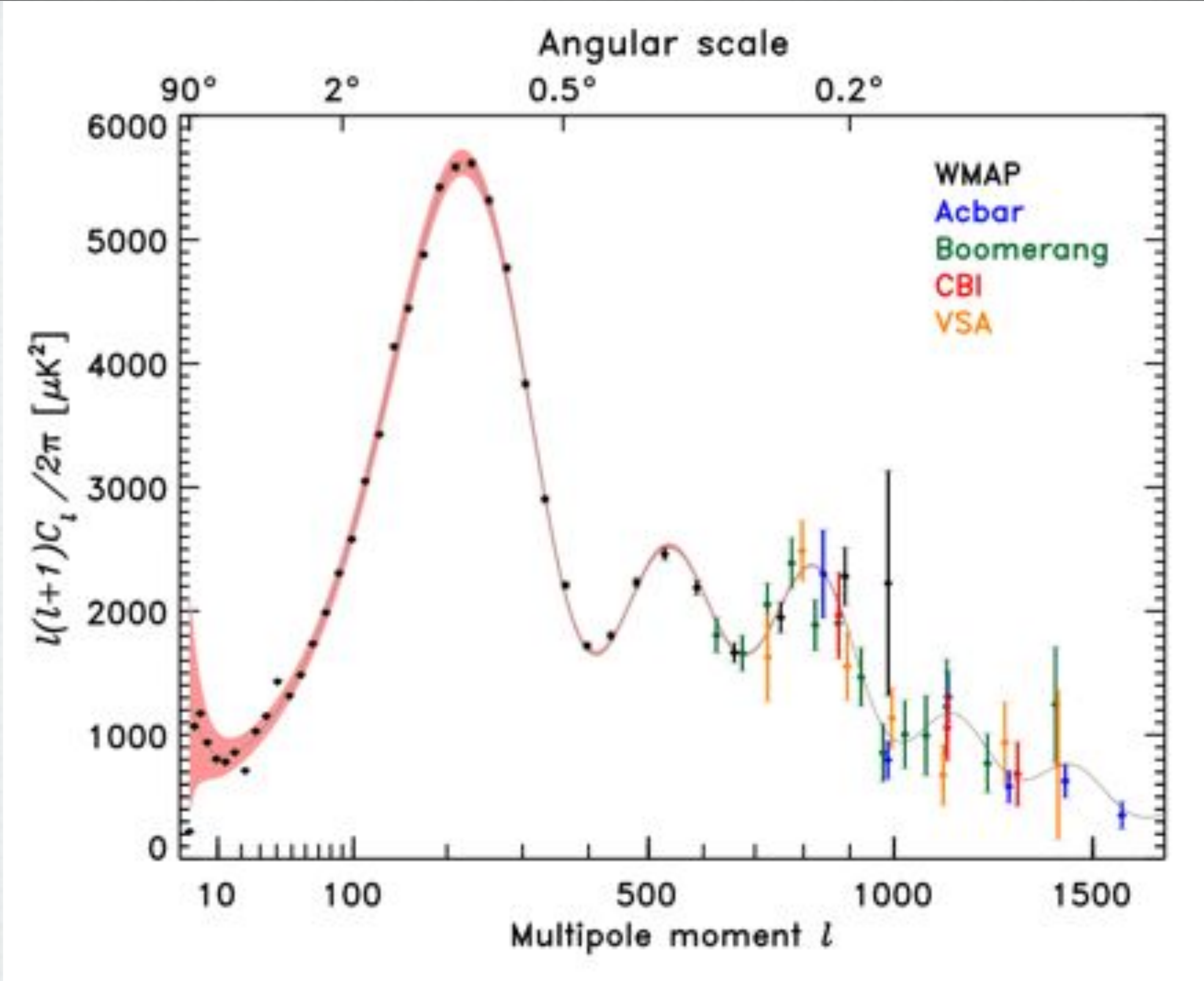
Background radiation released.

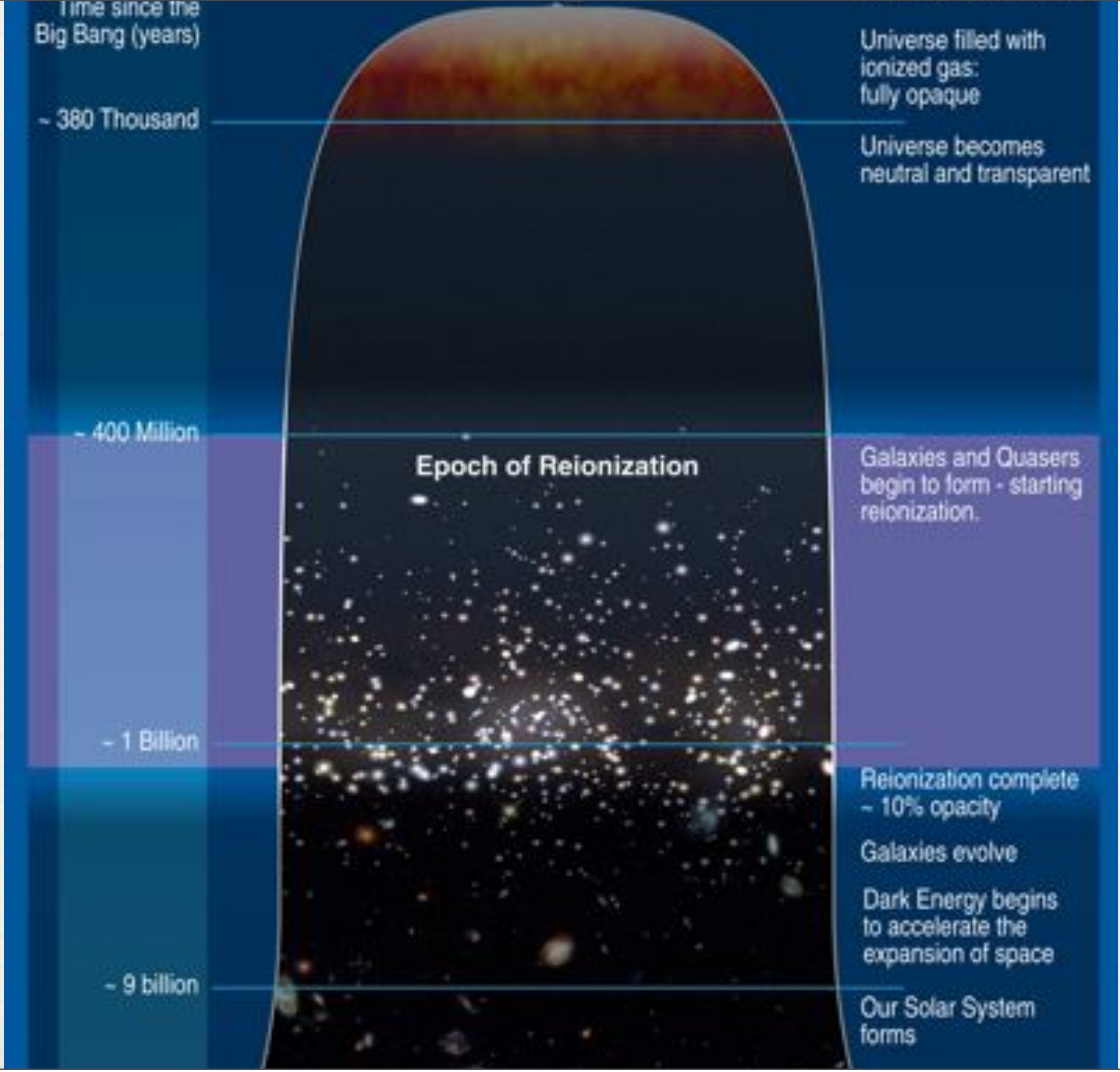



Animation courtesy of
NASA and WMAP


COBE
WMAP
PLANCK


7000
6000
5000
4000
3000
2000
1000
0
$\ell(\ell+1)C_\ell/2\pi$ (μK^2)
10
100
1000
Multipole ℓ
Acoustic Peaks
Damping Tail
Sachs–Wolfe Plateau
ISW Rise
Tensors


Angular scale
90°
2°
0.5°
0.2°
6000
5000
4000
3000
2000
1000
0
WMAP
Acbar
Boomerang
CBI
VSA
$l(l+1)C_l/2\pi$ [μK^2]
10
100
500
1000
1500
Multipole moment l

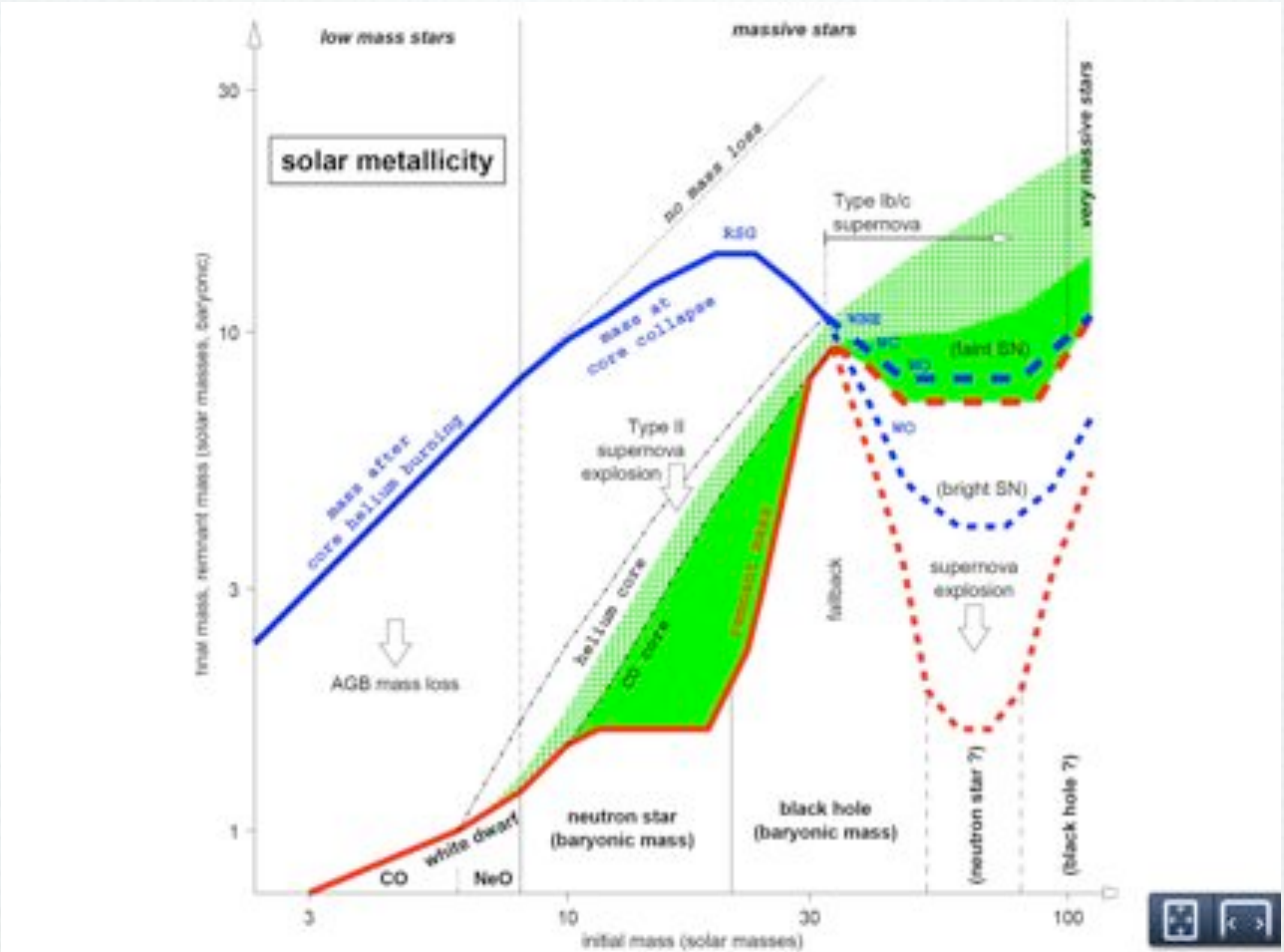

Time since the
Big Bang (years)
~ 380 Thousand
~ 400 Million
~ 1 Billion
~ 9 billion
Universe filled with
ionized gas:
fully opaque
Universe becomes
neutral and transparent
Epoch of Reionization
Galaxies and Quasers
begin to form - starting
reionization.
Reionization complete
~ 10% opacity
Galaxies evolve
Dark Energy begins
to accelerate the
expansion of space
Our Solar System
forms

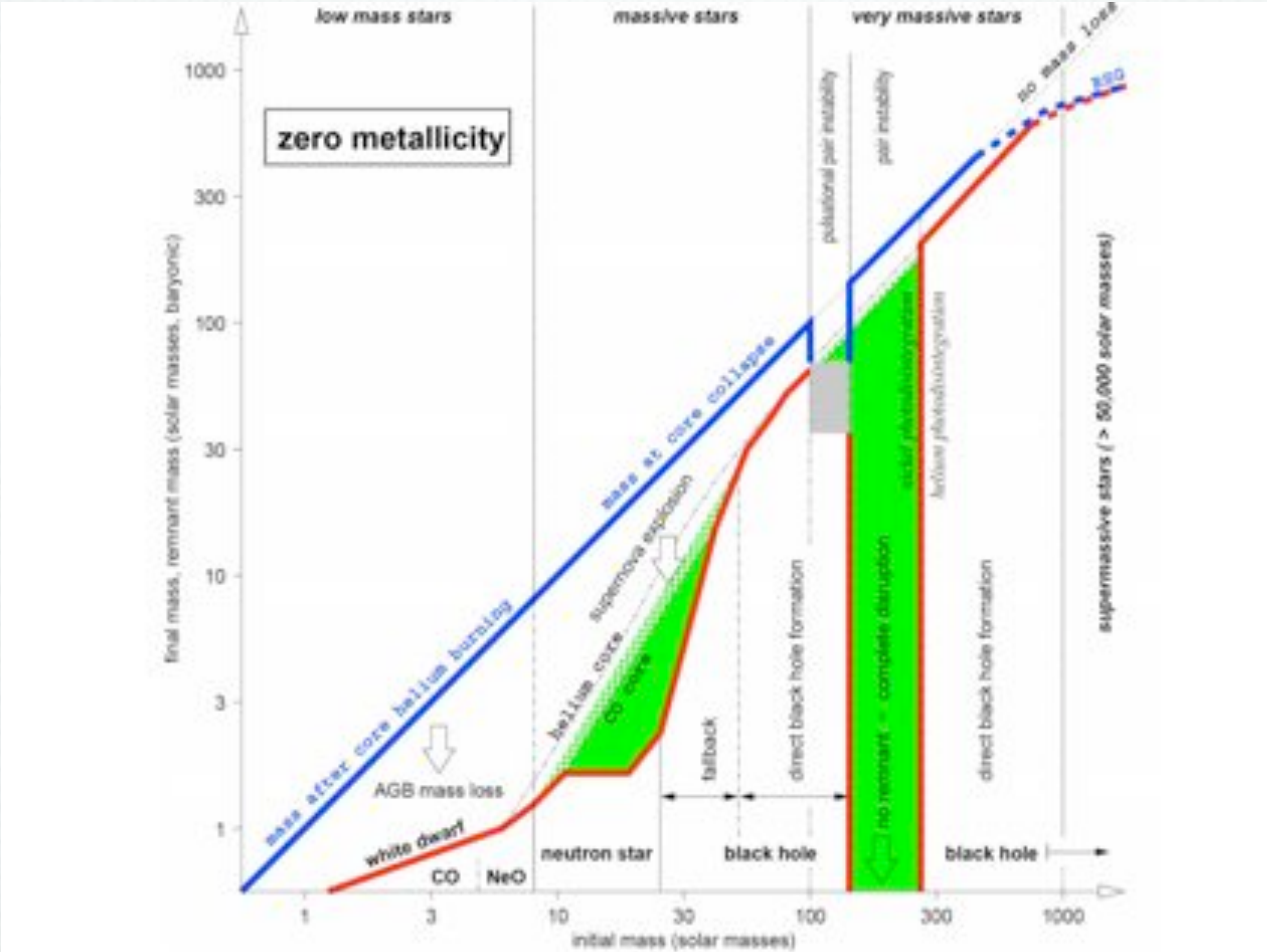
Then, all is dark, for awhile

* But not for too long
* Gravitationally over dense regions began to collapse
* Seeds of formation were regions of dark matter which did not interact with radiation
* Matter began Jean's collapse
* No metals to cool collapsing matter
* Collapsing clouds had to be bigger for gravity to overcome thermal pressure

First Stars

* Were bigger and primarily metal free, simulations and theory resulted in stars that could be up to 1000 times larger than our sun, today only up to about 100 times larger
* For a certain mass range some of these died in pair instability supernovae
  * can't happen today

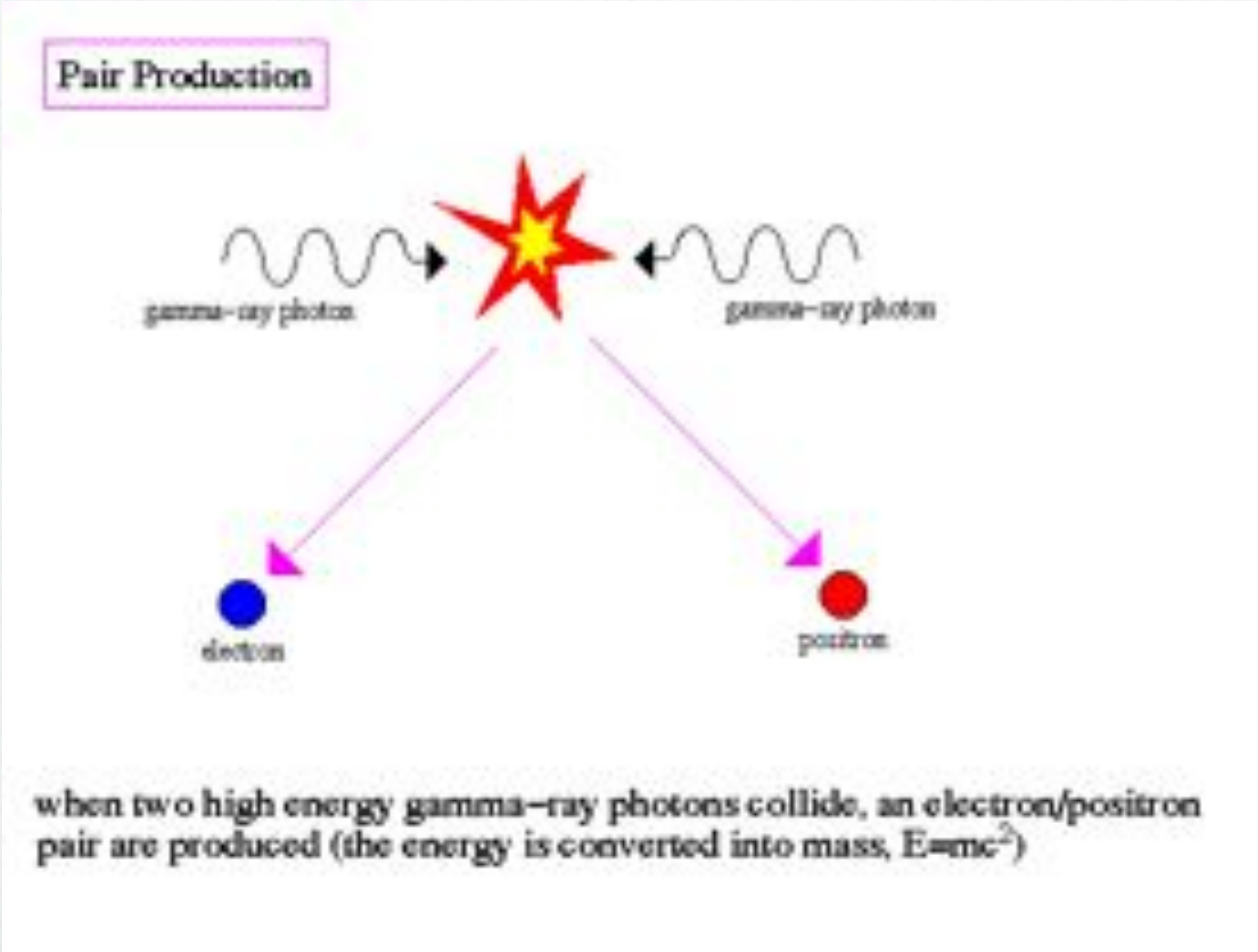

low mass stars
massive stars
very massive stars
solar metallicity
no mass loss
mass after
core helium burning
mass at
core collapse
RSG
Type Ib/c
supernova
WNE
WC
WO
(faint SN)
WO
(bright SN)
Type II
supernova
explosion
helium core
CO core
remnant mass
fallback
supernova
explosion
AGB mass loss
white dwarf
CO
NeO
neutron star
(baryonic mass)
black hole
(baryonic mass)
(neutron star ?)
(black hole ?)
final mass, remnant mass (solar masses, baryonic)
initial mass (solar masses)
30
10
3
1
3
10
30
100



low mass stars
massive stars
very massive stars
zero metallicity
1000
300
100
30
10
3
1
final mass, remnant mass (solar masses, baryonic)
mass after core helium burning
mass at core collapse
no mass loss
supernova explosion
helium core
CO core
AGB mass loss
white dwarf
CO
NeO
neutron star
black hole
black hole
fallback
direct black hole formation
pulsational pair instability
pair instability
no remnant - complete disruption
direct black hole formation
supermassive stars (/ > 50,000 solar masses)
1
3
10
30
100
300
1000
initial mass (solar masses)

Pair Instability Supernovae

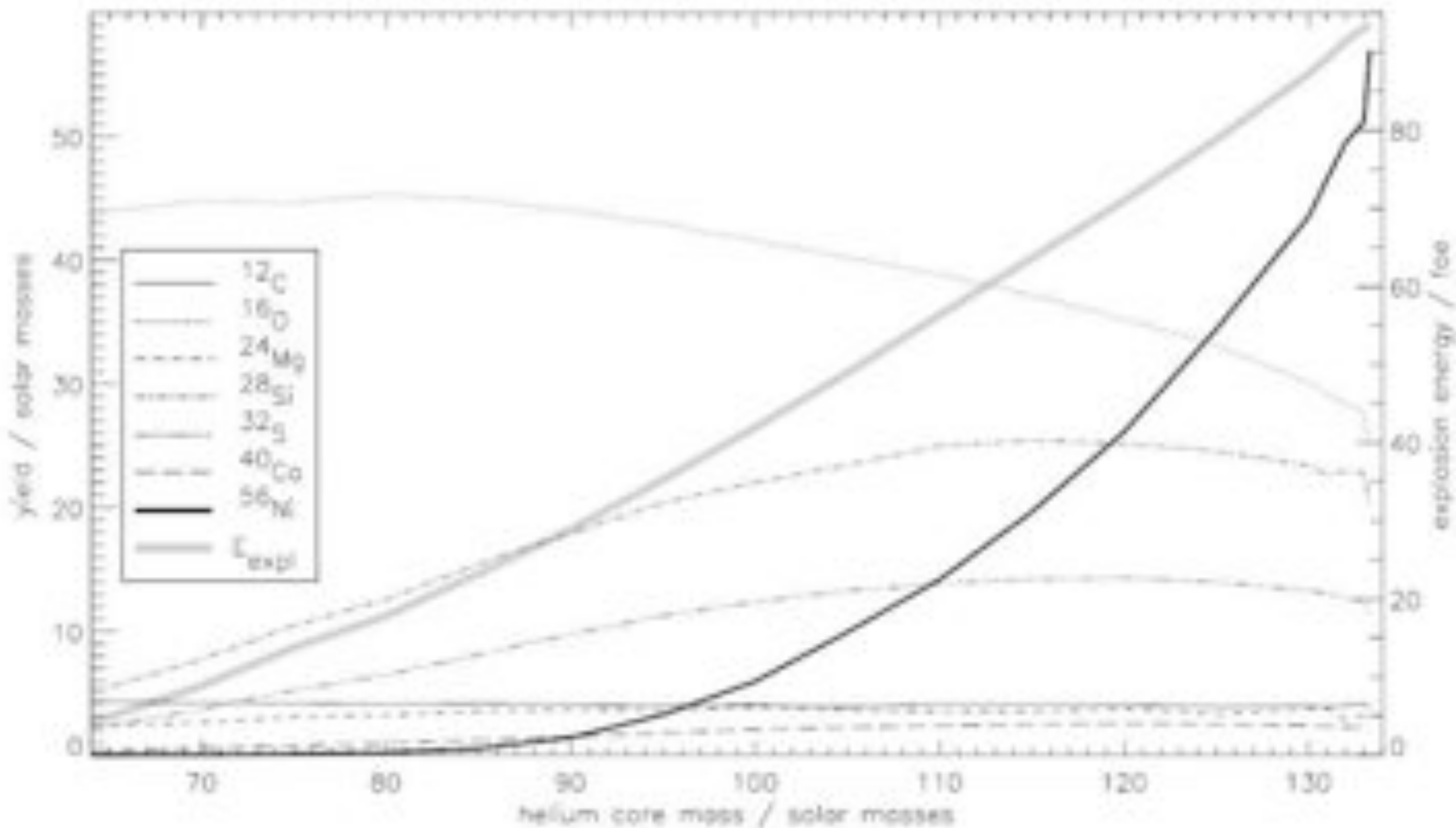
* 130-250 solar mass stars
* In sufficiently massive stars the gamma ray photons in the core can be energetic enough to create positron electron pairs via the reaction



Pair Instability Supernovae

* The creation of pairs decreases the luminosity of gamma ray photons needed to provide the pressure support versus gravitational collapse
* The star contracts, the core heats, more pairs are created, more pressure is removed
* Runaway process, instability
* The process proceeds rapidly and the star unbinds itself leaving no remnant
* All of the star blows up and nucleosynthesis creates the first metals at a high yield

Nucleosynthetic Yields



The universe now has metallicity

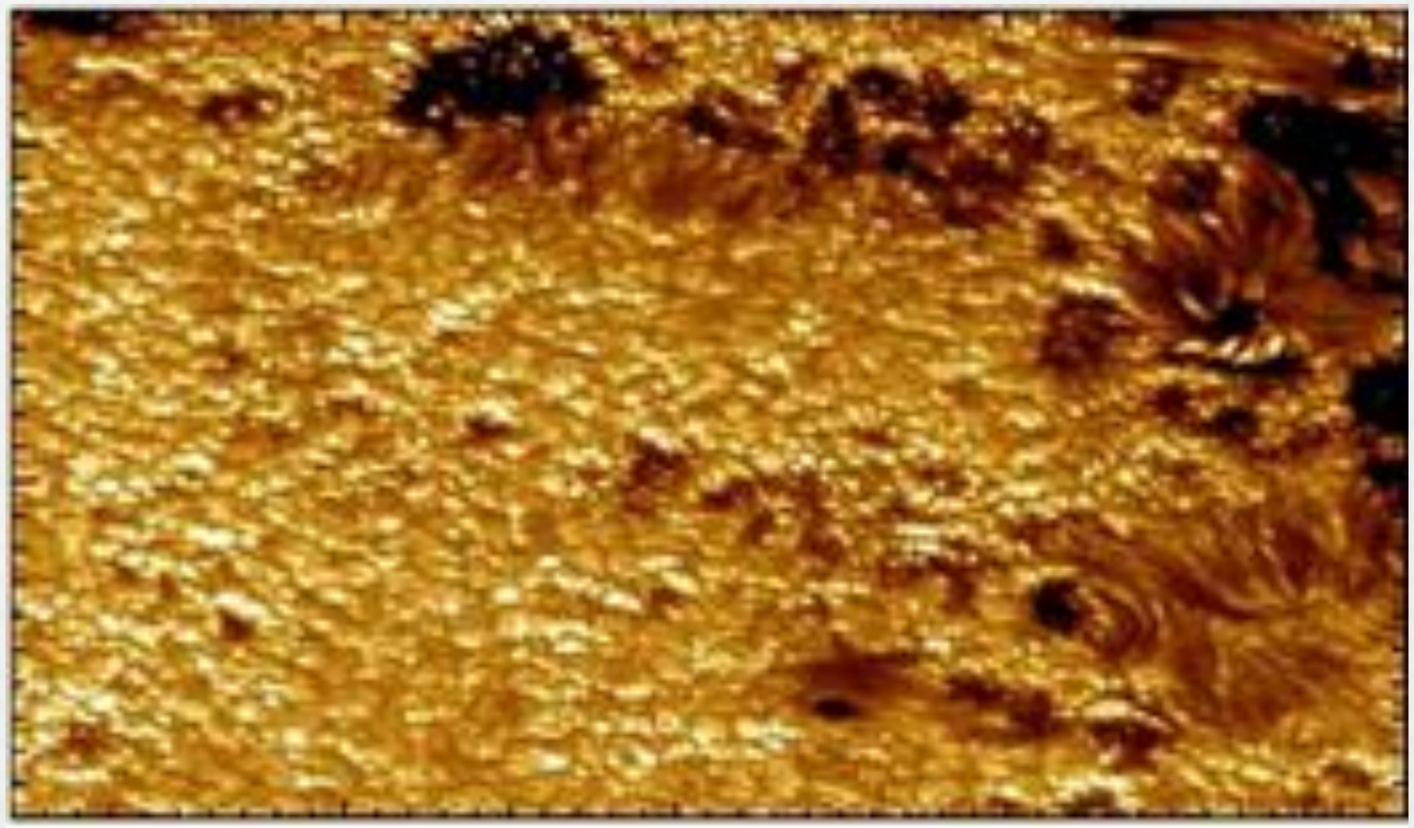
* Stars with metal form from reprocessed stuff
* Population 2 stars can form, metals can allow planet formation now
* Back to the class

Mixing length theory

* Or how efficient is convective energy transport
* Defined by free parameters, pure underlying physics from first principle are not known
* Really need to do hydrodynamical simulations to model

When Superadiabatic

* Convection carries energy
* How much?



Start with criteria for convection

$$|\frac{dT}{dr}|^{(s)} - |\frac{dT}{dr}|^{(b)} > 0$$

or, since gradients are negative

$$\frac{dT}{dr}^{(b)} - \frac{dT}{dr}^{(s)} > 0$$

Assuming the bubble moves adiabatically

$$\frac{dT}{dr}^{(b)} = \frac{dT}{dr}_{adiabatic}$$

And

$$\frac{dT}{dr}^{(s)} = \frac{dT}{dr}_{actual}$$

after the bubble has moved a distance dr the temperature of the bubble will exceed that of the surroundings by

$$\delta T = (\frac{dT}{dr}|_{adiabatic} - \frac{dT}{dr}|_{actual})dr = \delta(\frac{dT}{dr})dr$$

The bubble travels a length l before depositing its heat into the surroundings, this length is not known from first principles

$$l = \alpha H_p$$

Where Hp is the pressure scale height and l is the mixing length, α is a free parameter generally between .5 and 3

The excess heat, per unit volume dissipated into the surroundings is given by

$$\delta q = (C_p \delta T)\rho$$

Substituting l for dr and multiplying by the average velocity of the bubble gives the heat flux or energy per unit time per unit area carried by the bubble

$$F_c = \delta q \bar{v}_c = (C_p \delta T) \rho \bar{v}_c$$

Where ρv is the mass flux

We may find the average velocity by determining the net force on the bubble, using the ideal gas law, and assuming constant mean molecular weight

$$\delta P = \frac{P}{\rho}\delta\rho + \frac{P}{T}\delta T$$

The deltas indicate the difference between bubble and surrounding gas which is zero for pressure

$$\delta P = P^{(b)} - P^{(s)} = 0$$

Giving

$$\delta\rho = -\frac{\rho}{T}\delta T$$

or, on recalling the net force is simply the bouyancy force minus the weight of the bubble

$$f_{net} = -g\delta\rho = -g(\rho^b - \rho^s)$$

So

$$f_{net} = \frac{\rho g}{T}\delta T$$

We assumed that the initial temperature difference between the bubble and its surroundings was almost zero and average between the initial and final temperature difference achieved after a bubble has moved a length l

$$< f_{net} >= \frac{1}{2}\frac{\rho g}{T}\delta T_f$$

From basic physics, the change in kinetic energy is equal to the work done on an object. The net work done by the buoyant force is then

$$\frac{1}{2}\rho v_f^2 =< f_{net} > l = \frac{1}{2}\frac{\rho g}{T}\delta T_f l$$

We choose an average velocity given by βv^2 where β ranges from 0 to 1, substitute the value for δT from above and arrive at an average convective velocity of

$$\bar{v}_c = (\frac{2\beta < f_{net} > l}{\rho})^{1/2}$$

using dr = l and our expression for δT

$$\bar{v}_c = (\frac{\beta g}{T})^{1/2}[\delta(\frac{dT}{dr})]^{1/2} l$$

$$= \beta^{1/2}(\frac{T}{g})^{1/2}(\frac{k_B}{\mu m_h})[\delta\frac{dT}{dr}]^{1/2}\alpha$$

Where we have used

$$\frac{1}{H_p} = -\frac{1}{P}\frac{dP}{dr} = -\frac{1}{P}(-1)\rho g$$

The ideal gas law

$$P = \frac{\rho k_B T}{\mu m_h}$$

and $l = \alpha H_p$

After some manipulation

$$F_c = \delta q \bar{v}_c = (C_p \delta T) \rho \bar{v}_c$$

$$= \rho C_p (\frac{k_B}{\mu m_h})^2 (\frac{T}{g})^{3/2} \beta^{1/2} [\delta(\frac{dT}{dr})]^{3/2} \alpha^2$$

Which is minimally effected by β but is more strongly effected by α, both free parameters in the theory

To evaluate this we still need to know the difference between the temperature gradients and make the simplifying assumption that all of the flux is carried by convection

$$F_c = \frac{L_r}{4\pi r^2}$$

Which results in

$$\delta(\frac{dT}{dr}) = [\frac{L_r}{4\pi r^2}\frac{1}{\rho C_p \alpha^2}(\frac{\mu m_H}{k_B})^2(\frac{g}{T})^{3/2}\beta^{-1/2}]^{2/3}$$

To determine how superadiabatic the temperature gradient must be to carry all of the flux by convection we divide the difference in temperature gradients by the adiabatic temperature gradient

$$\frac{\delta(\frac{dT}{dr})}{|\frac{dT}{dr}|_{adiabatic}} = (\frac{L_r}{4\pi r^2})^{2/3} C_p^{1/3}\rho^{-2/3}\alpha^{-4/3}(\frac{\mu m_h}{k_B})^{4/3}T^{-1}\beta^{-1/3}$$

For parameters specific to the sun we find

$$\delta(\frac{dT}{dr}) = 6.7\times 10^{-9} Km^{-1}$$

$$|\frac{dT}{dr}|_{adiabatic} = .015 Km^{-1}$$

$$\frac{\delta(\frac{dT}{dr})}{|\frac{dT}{dr}|_{adiabatic}} = 4.4\times 10^{-7}$$

For stellar parameters the gradient difference needs to be tiny for convection to dominate

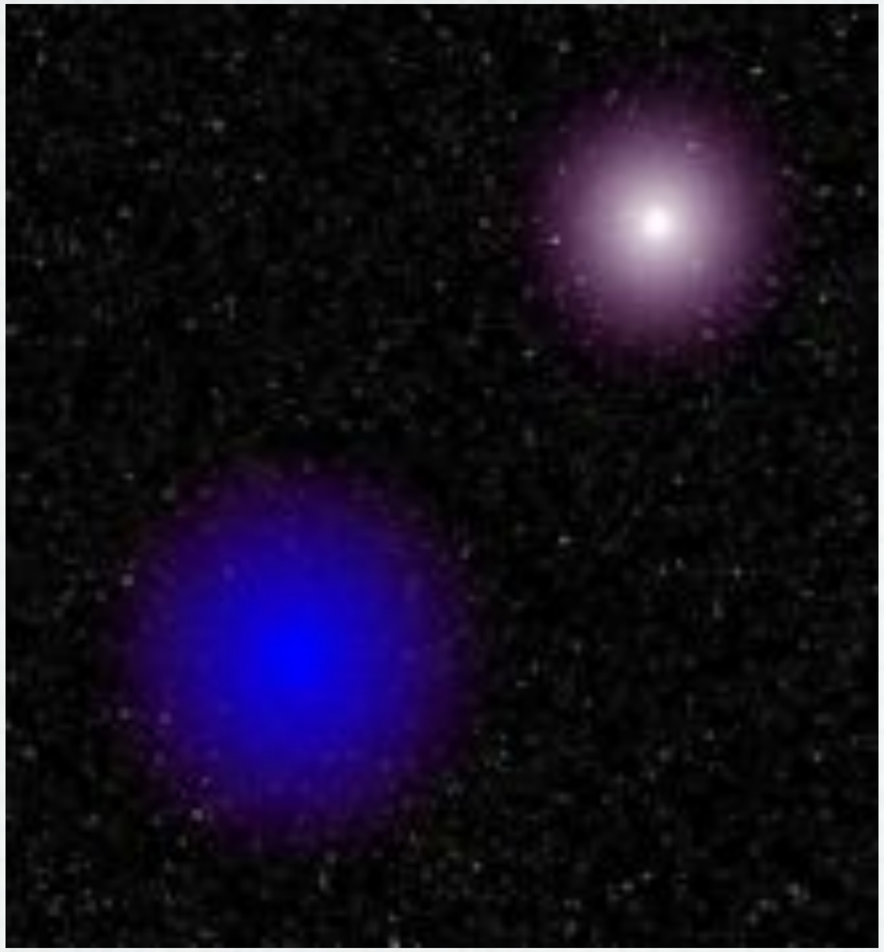
Variable Stars

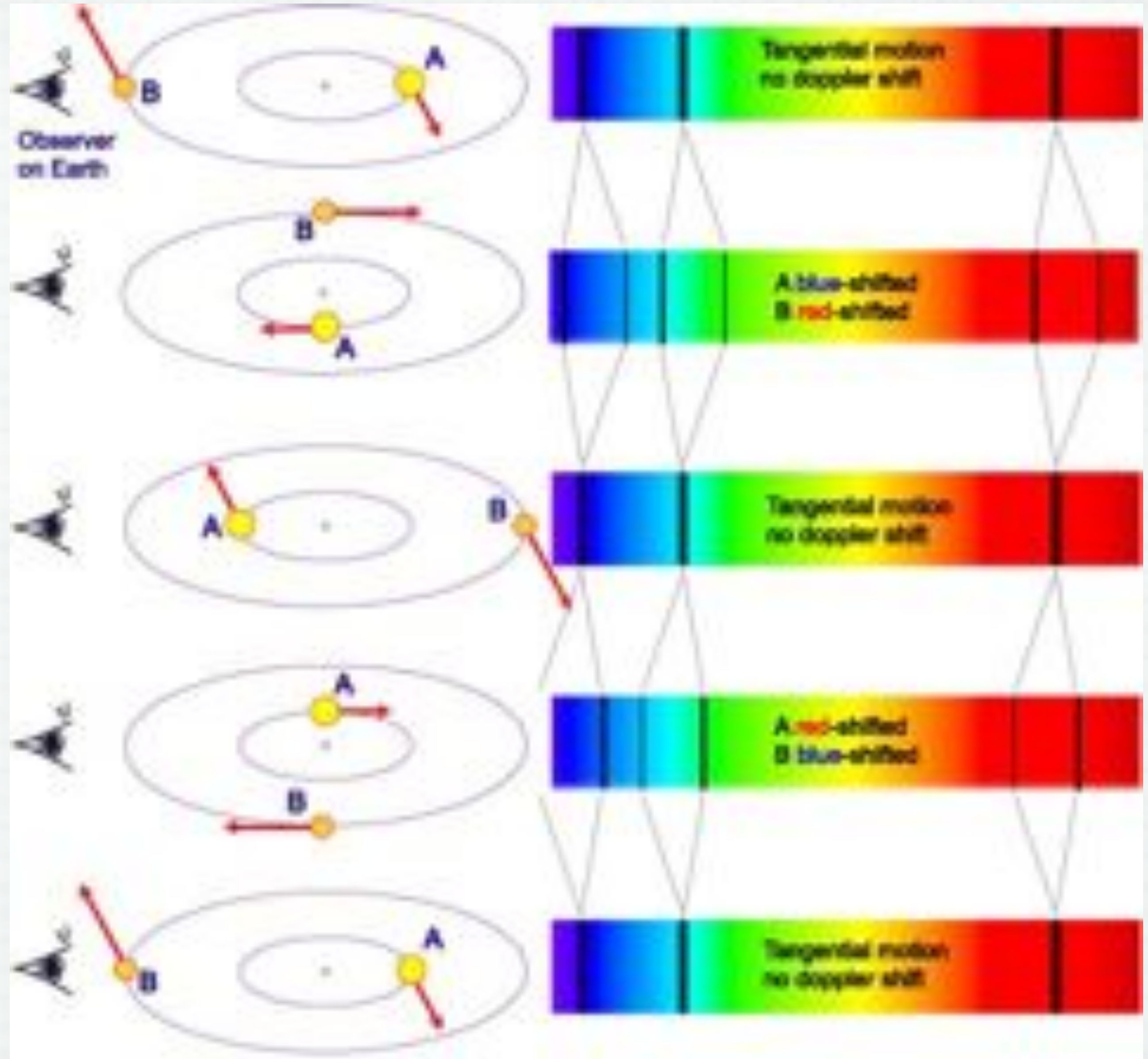
* Rotating
* Pulsating
* Eruptive
* Explosive

- Some act as standard candles
- This is very important in astrophysics
- Allows for distance measurements
- Need these for cosmology

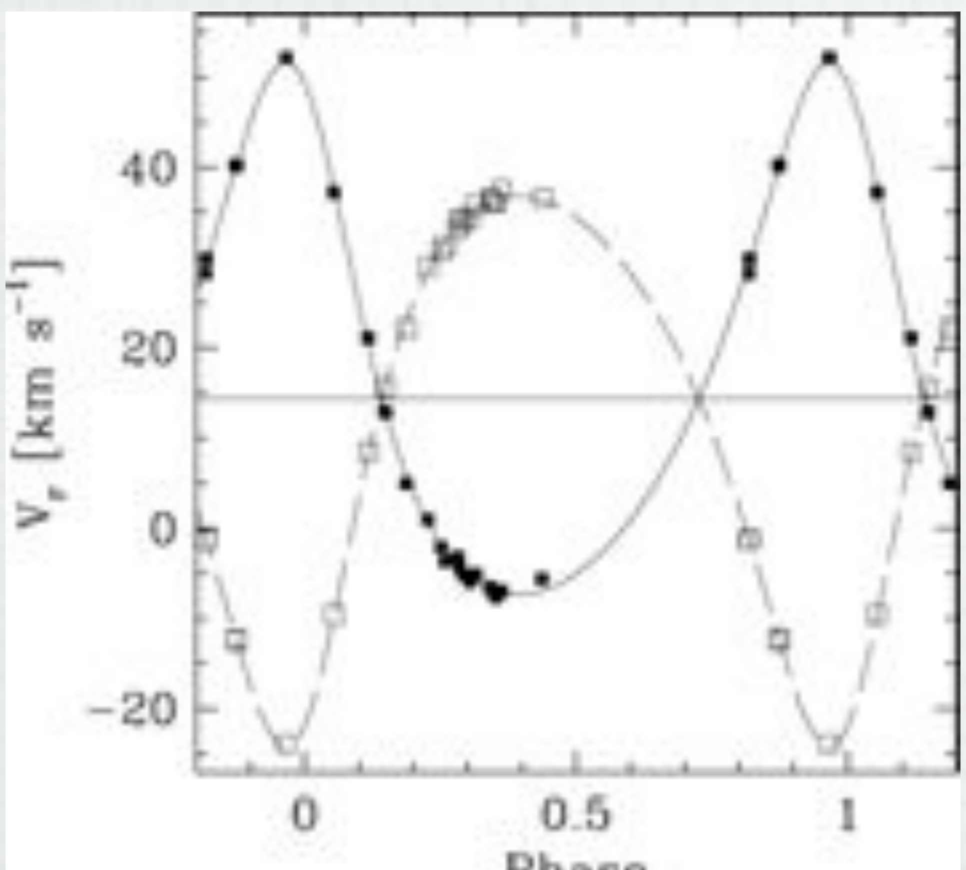
Rotating Variability

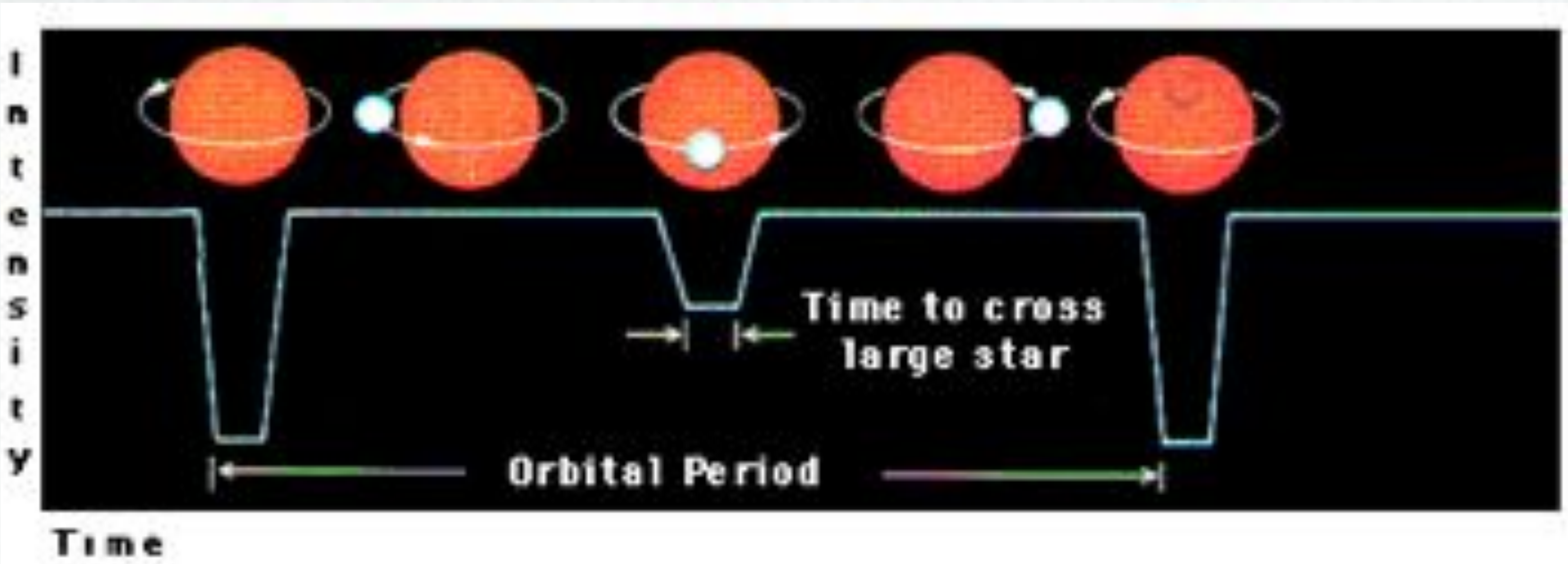
* Generally binaries
  * visibly resolvable
  * Spectroscopic (doppler shift tugs spectra back and forth)
  * eclipsing binaries
  * Astrometric

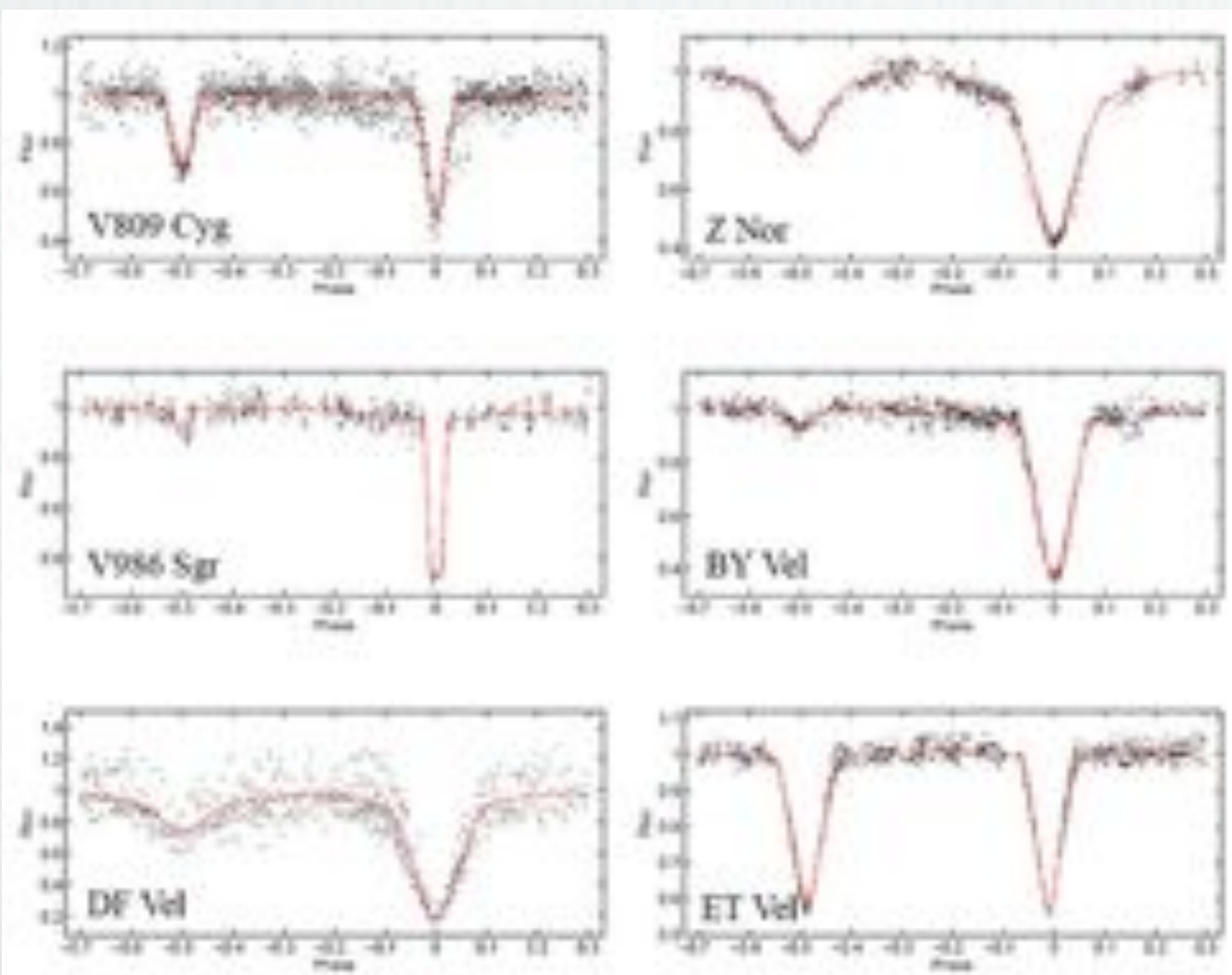


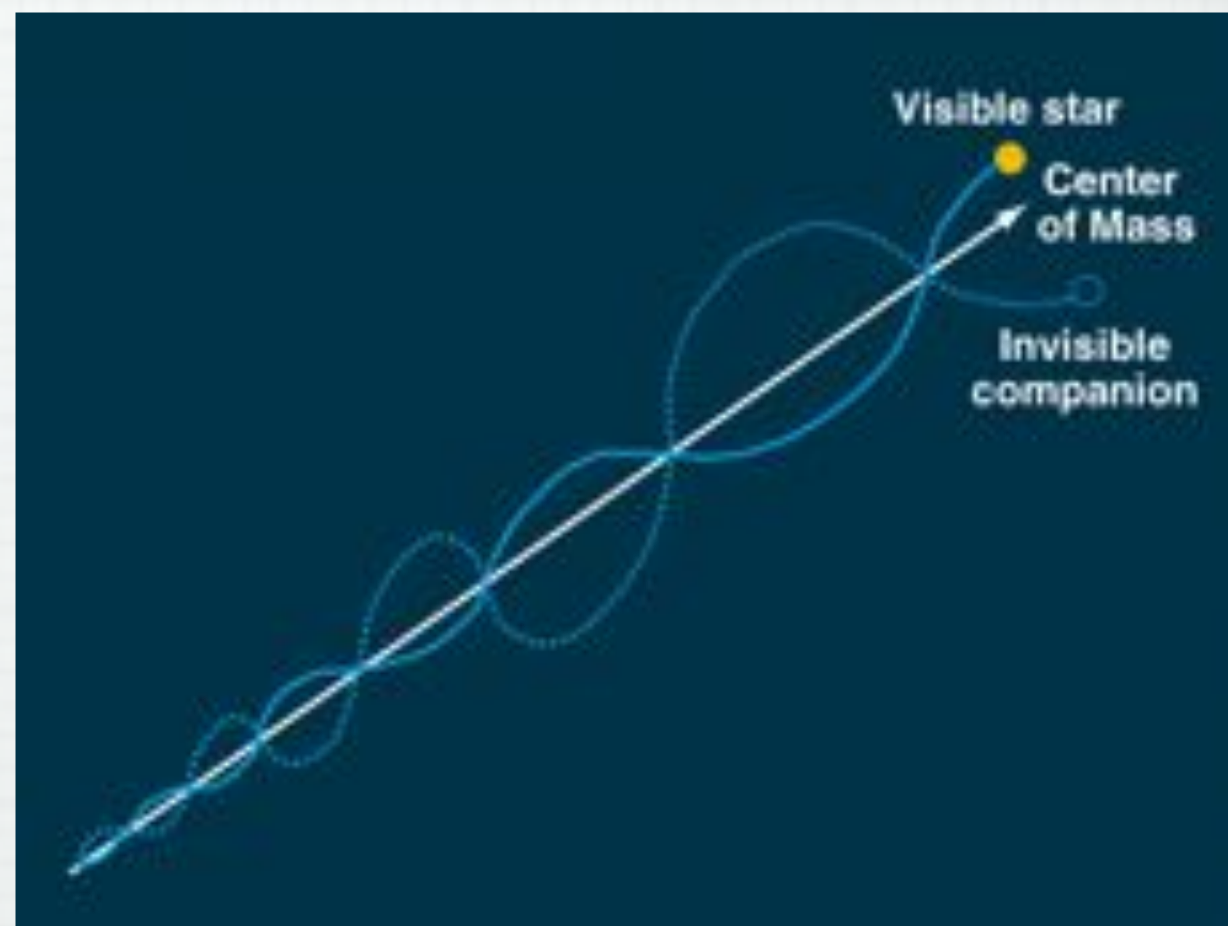


A Spectroscopic Binary System
High-mass star A and lower-mass B orbit around a common centre of mass. The observed combined spectrum shows periodic splitting and shifting of spectral lines. The amount of shift is a function of the alignment of the system relative to us and the orbital speed of the stars.



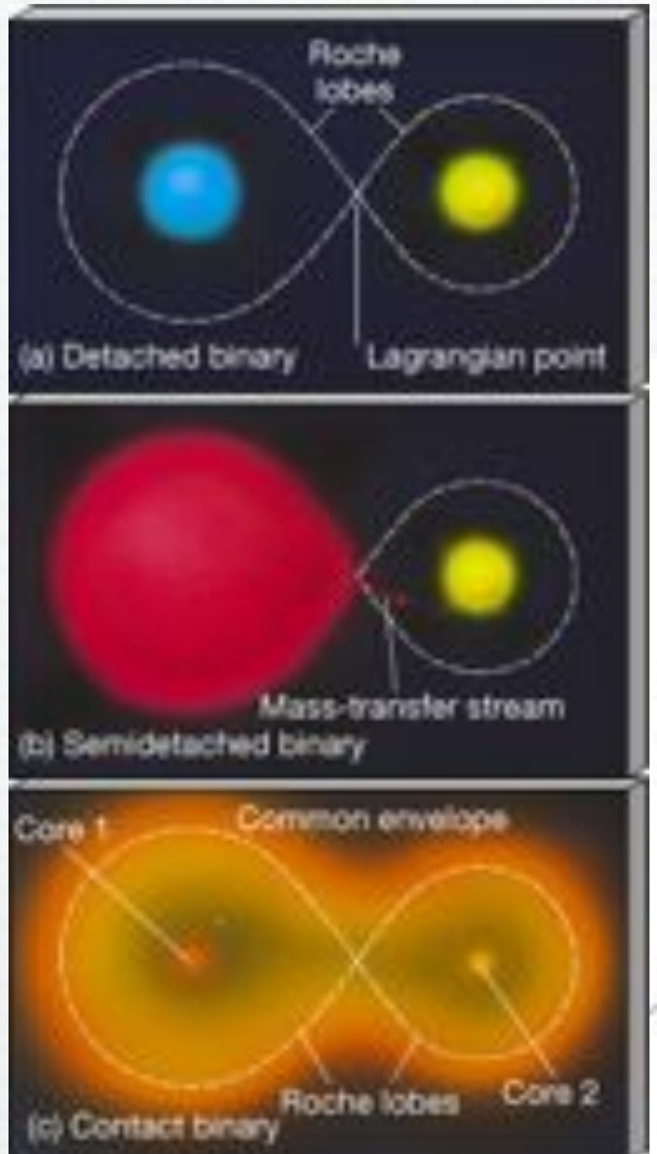

Intensity
Time to cross
large star
Orbital Period
Time

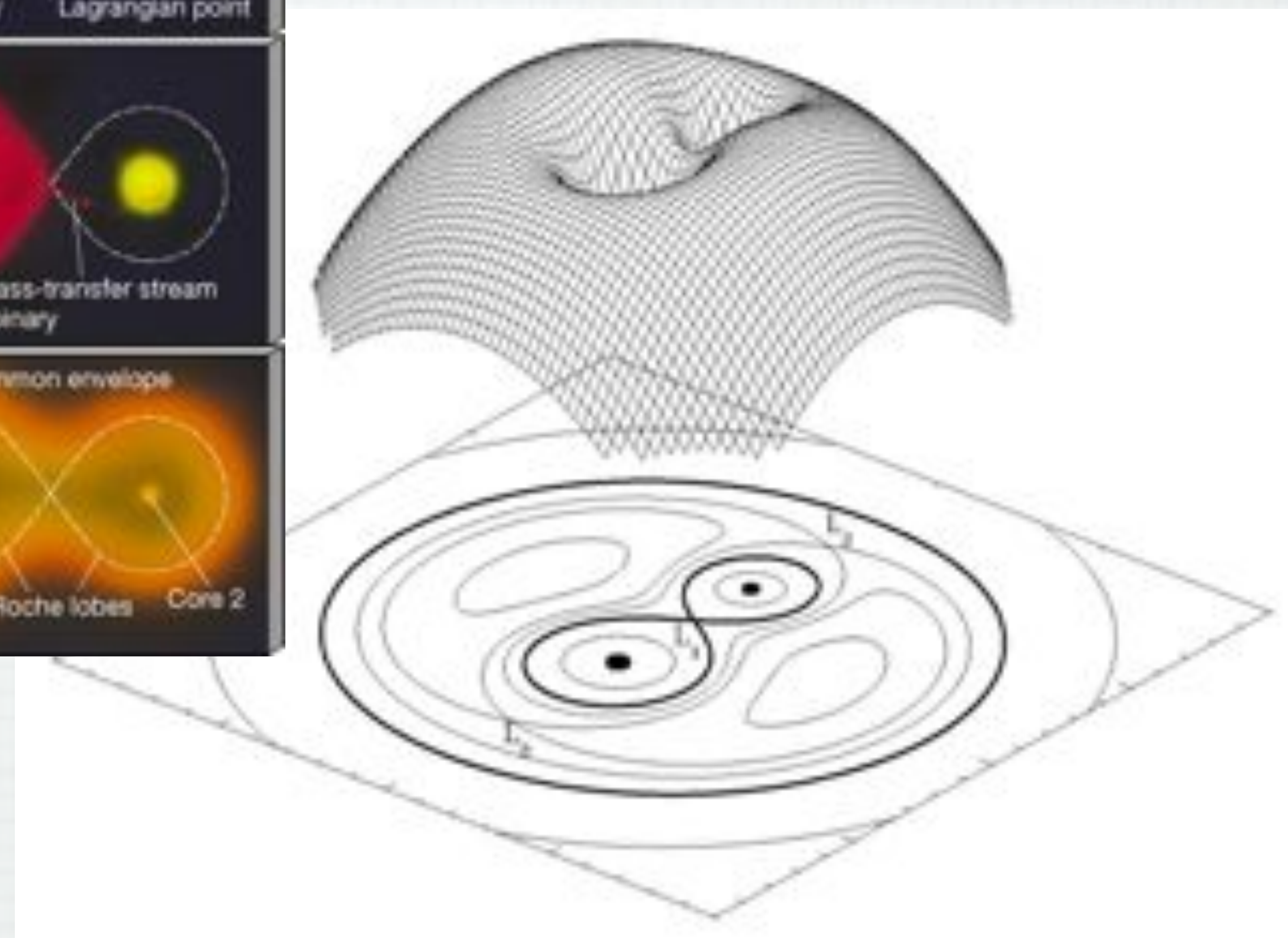

V809 Cyg
Z Nor
V986 Sgr
BY Vel
DF Vel
ET Vel


Visible star
Center
of Mass
Invisible
companion

Eruptive and Explosive

* Periodic explosions ranging from small outbursts to outbursts that unbind the star
* Generally due to binary systems where the evolution allows mass transfer
* Roche Lobe filled or not

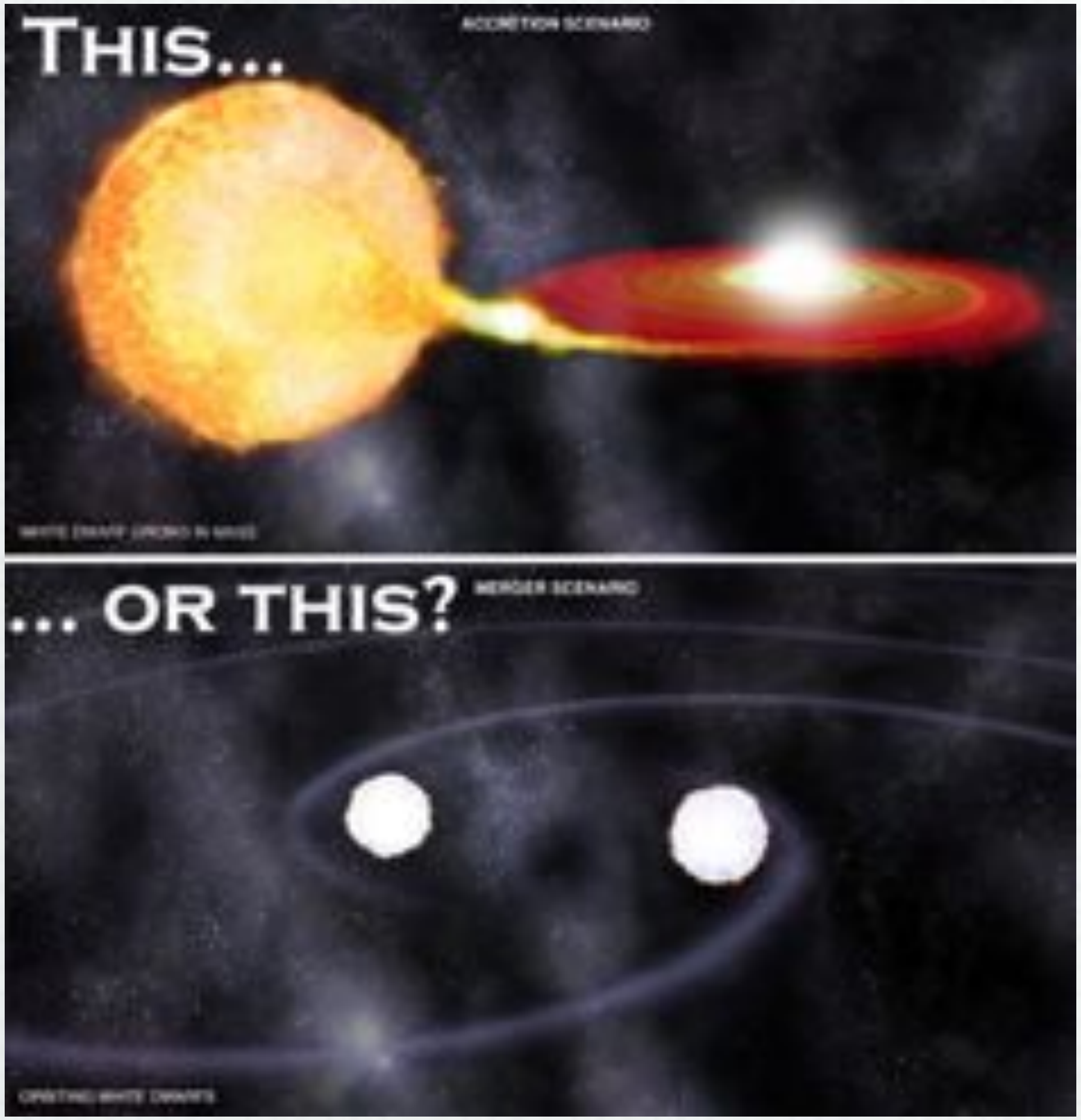




Huge Variety

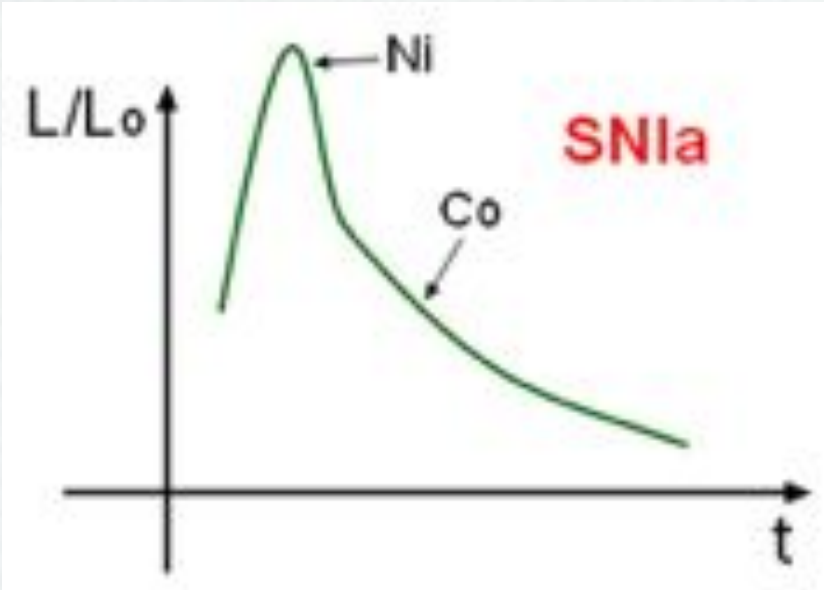
* http://en.wikipedia.org/wiki/Cataclysmic_variable_star
* Type 1a supernovae - standard candles, white dwarf accretes mass from companion until it reaches a critical mass then explodes
* Light curve is common to all, see light curve, know distance

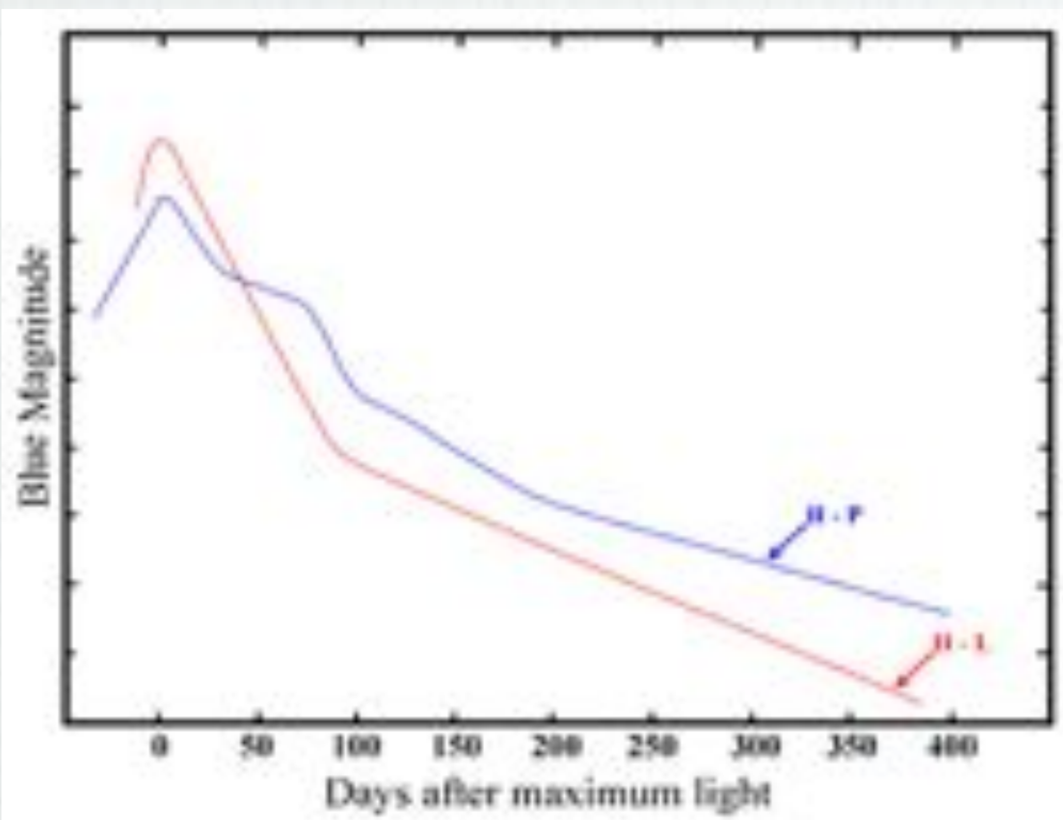
Type 1a supernovae

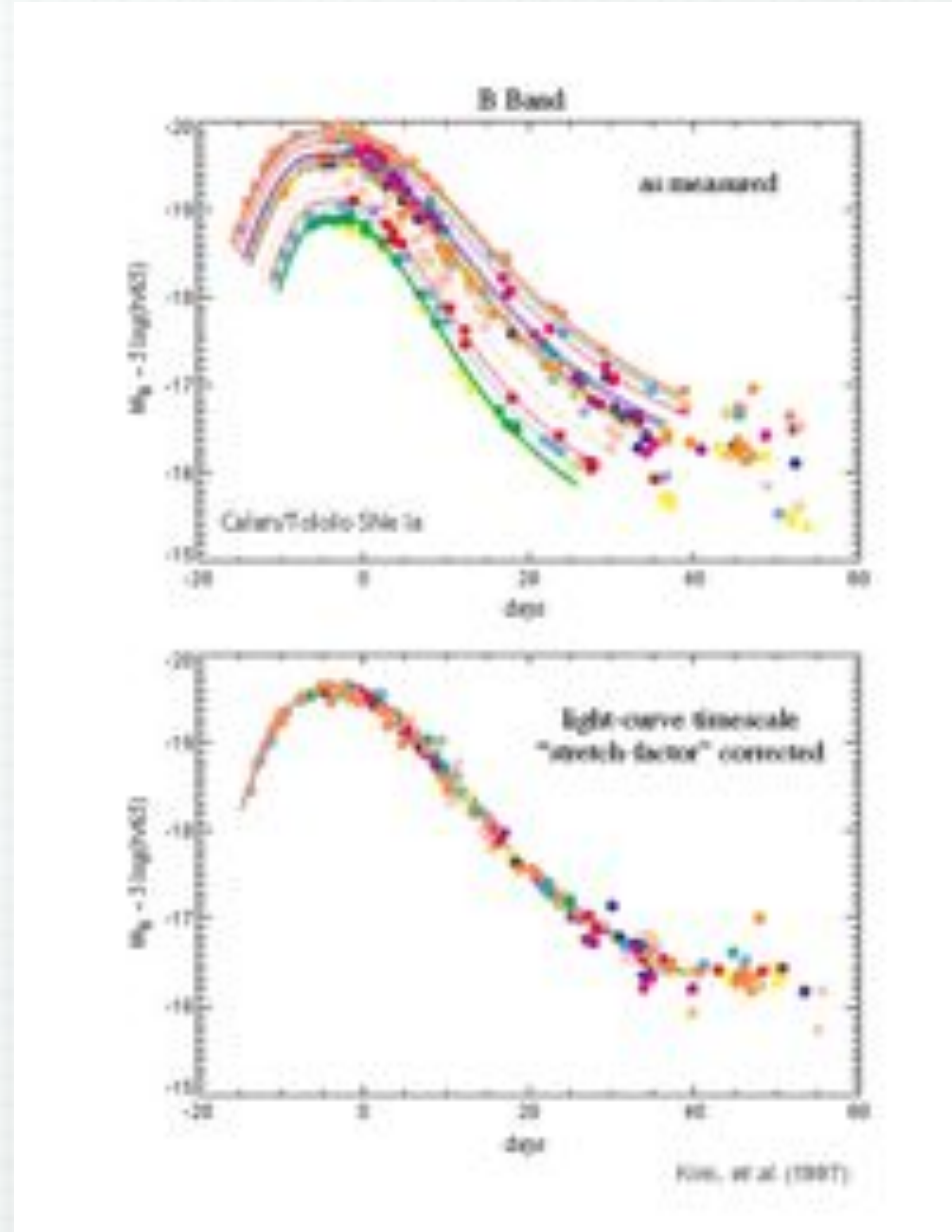


http://flash.uchicago.edu/site/movies/

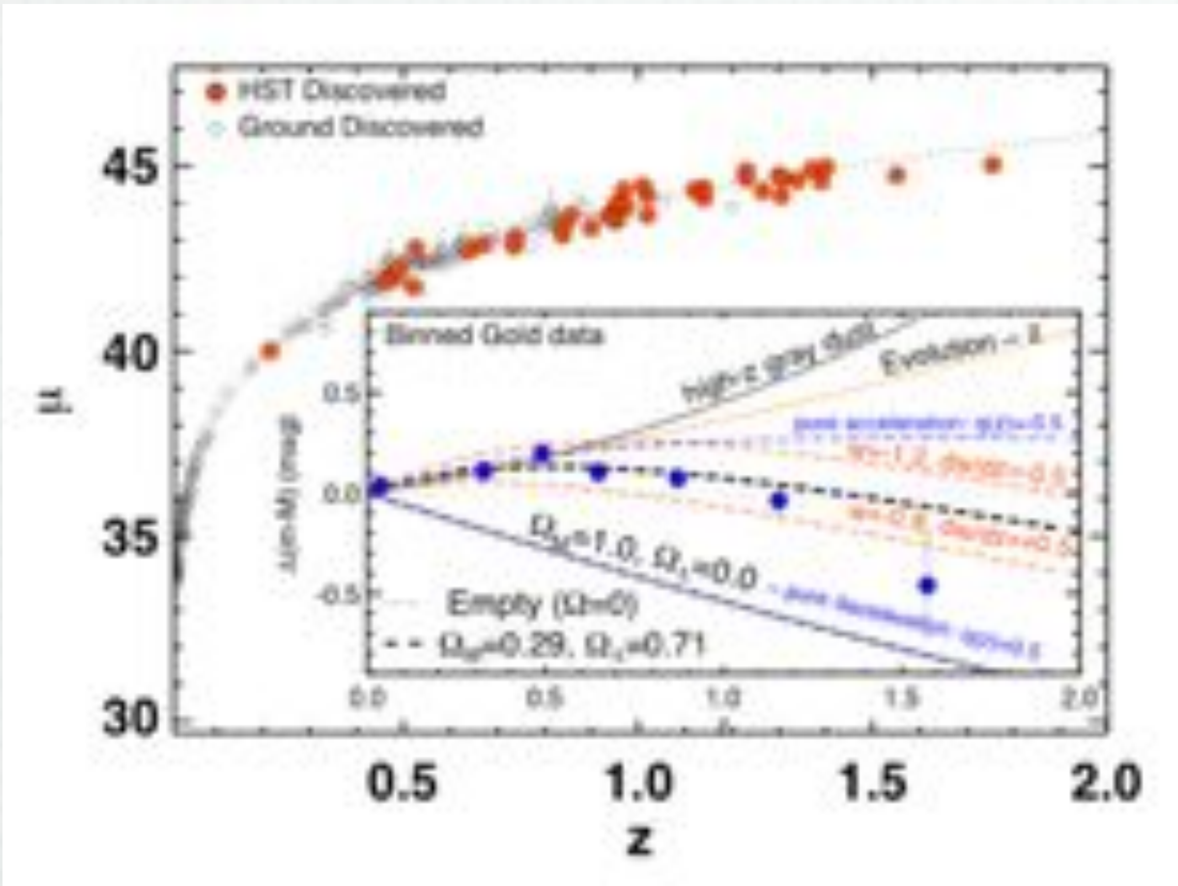
Type 1a supernovae

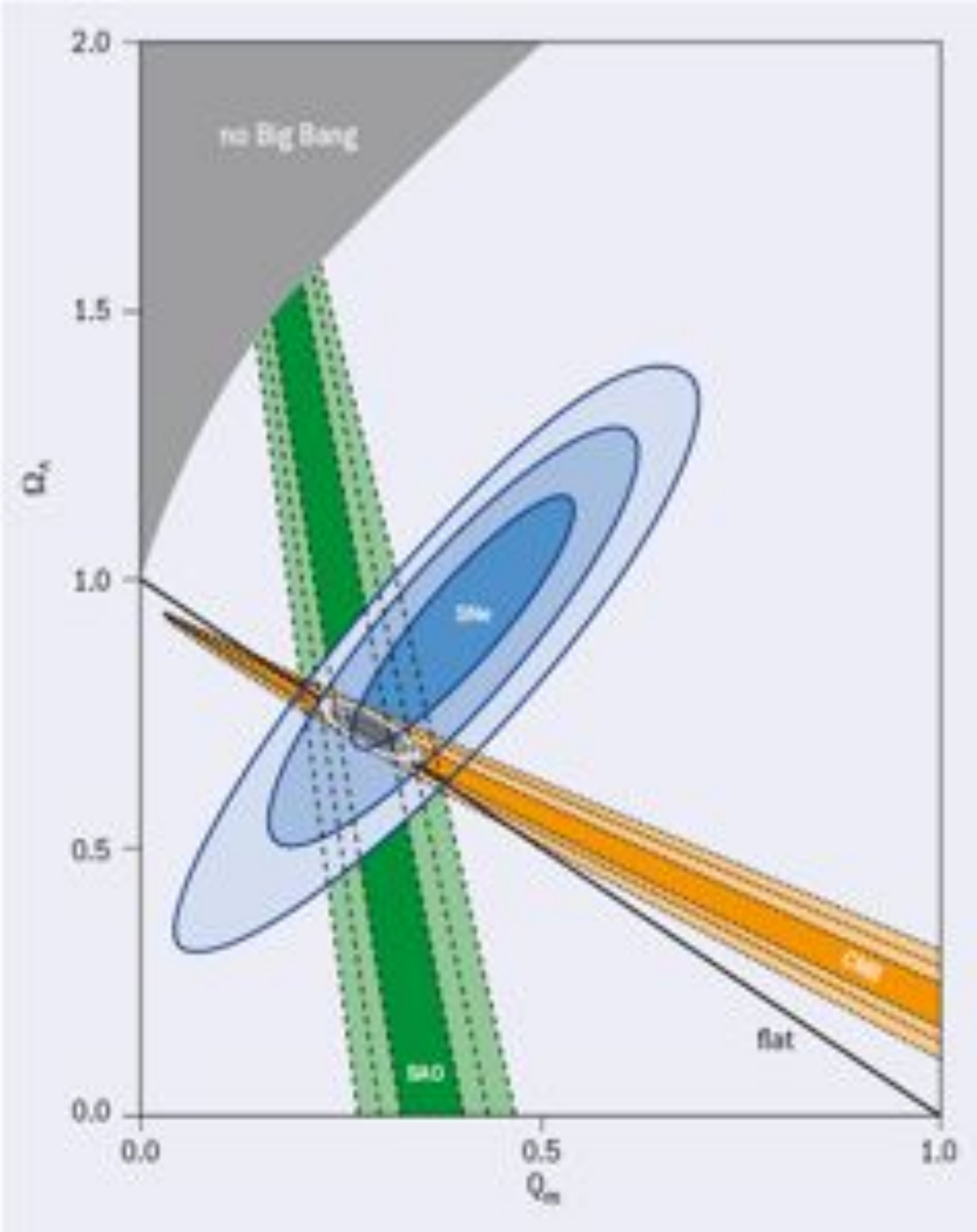






Dark Energy


HST Discovered
Ground Discovered
Binned Gold data
Empty (Ω=0)
Ω_M=0.29, Ω_Λ=0.71
Ω_M=1.0, Ω_Λ=0.0
Evolution ~ z
μ
z
45
40
35
30
0.5
1.0
1.5
2.0


no Big Bang
SNe
CMB
BAO
flat
Ω_Λ
Ω_m
2.0
1.5
1.0
0.5
0.0

Pulsating Variables

* Intrinsically out of equilibrium stars
* Surface radius or temperature periodically changes as the thermostat of the star is altered
* For some, this pulsation period allows a measure of the stars intrinsic brightness
* Standard Candles

Pulsating Variables

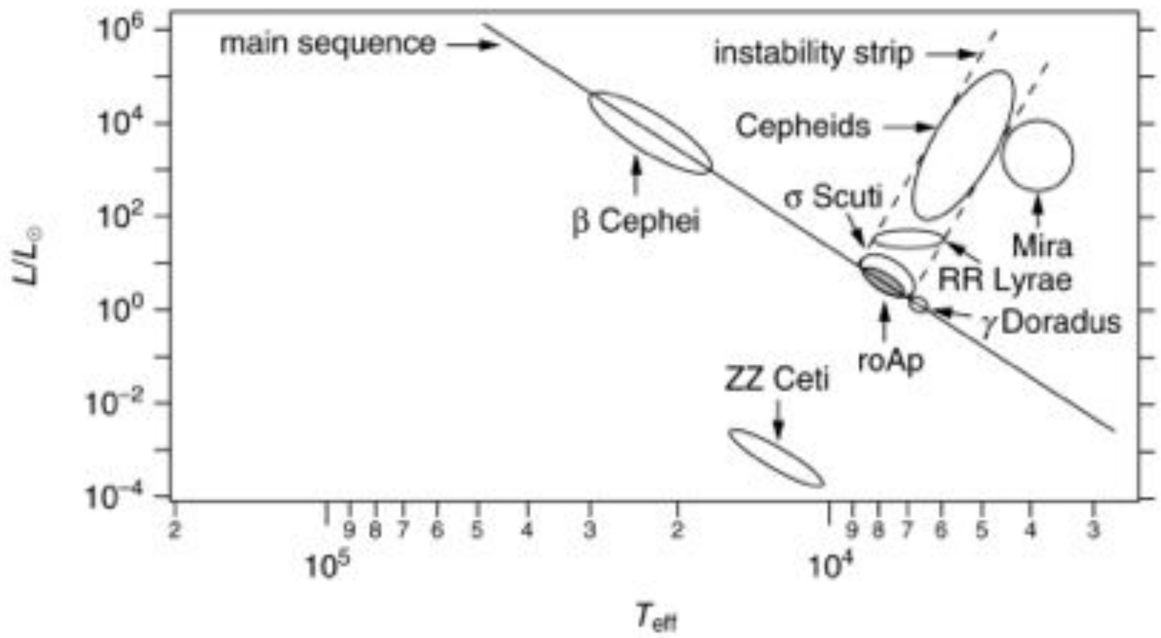


Figure 5.11 Approximate position of several types of pulsating stars in the H–R diagram. Also shown on the figure is the position of the main sequence (solid curve) and the instability strip (region between the dashed lines). Adapted with permission from Jörgen Christensen-Dalsgaard (private communication).

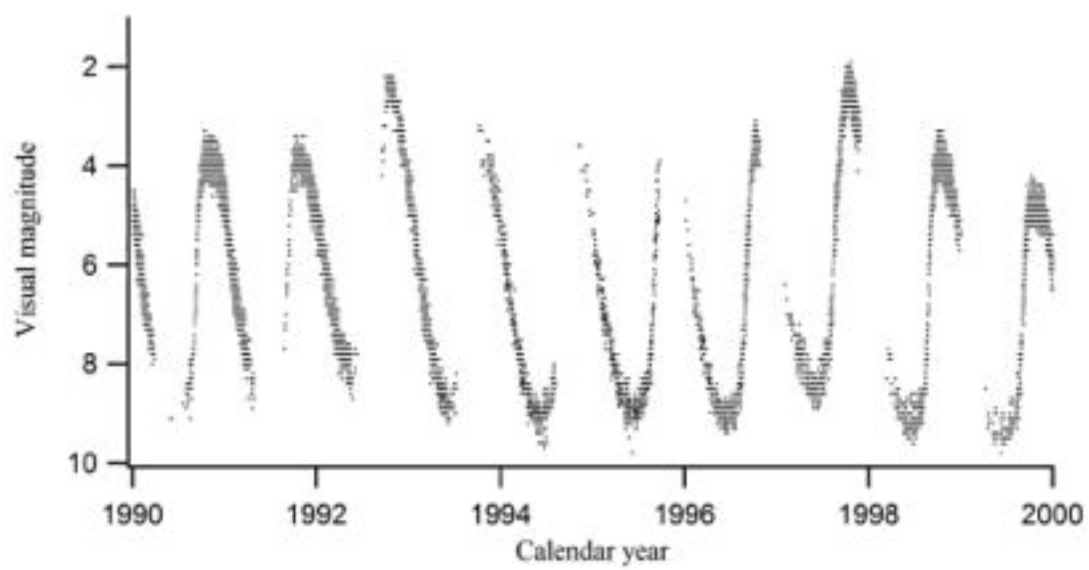
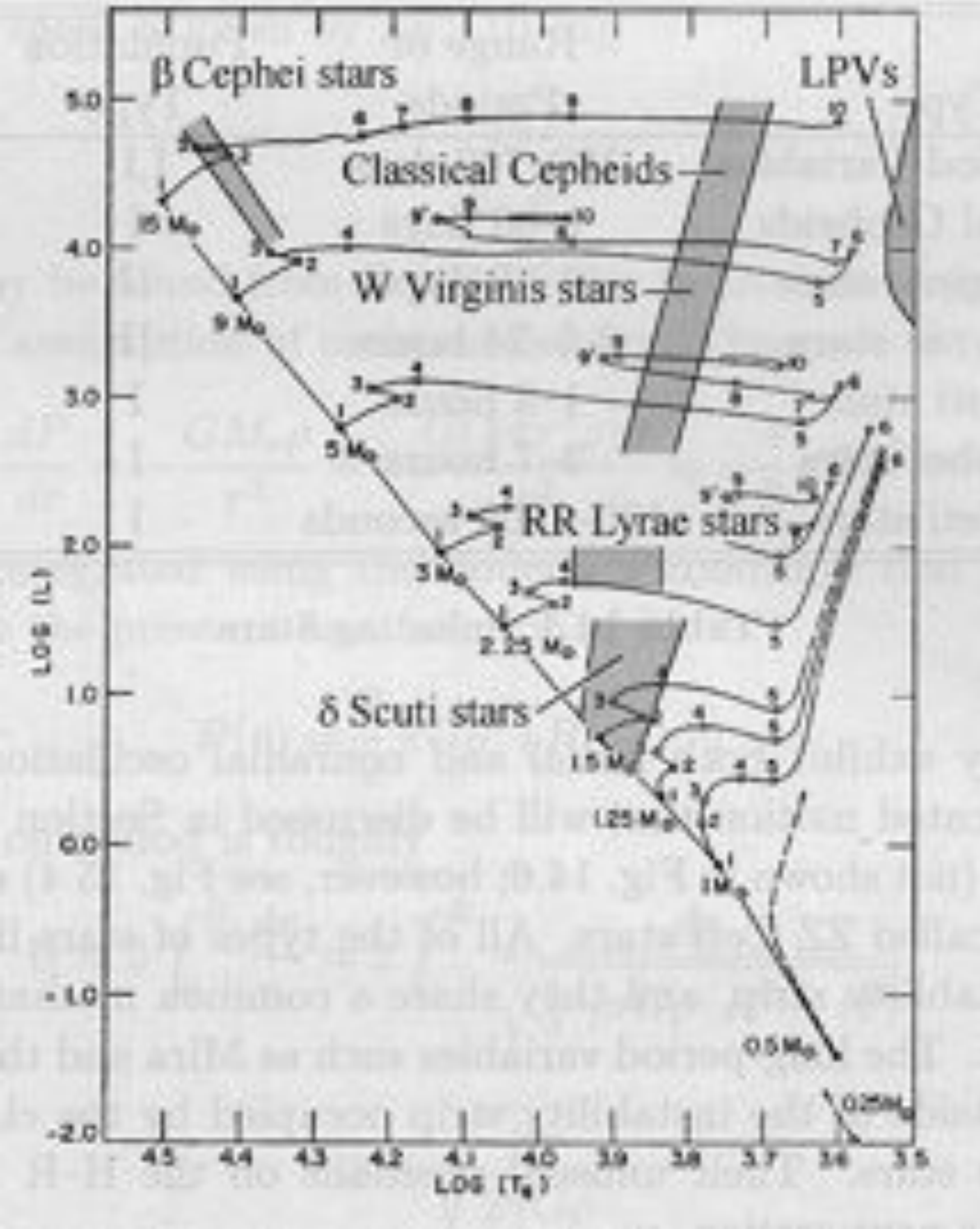


Figure 5.12 The observed visual magnitude for the prototype star for Mira variables, Omicron Ceti from 1990 to 2000. This star has a period of approximately 332 days, $M_* \approx 0.7\ M_\odot$ and $T_{eff} \approx 3000$ K. Data courtesy of the American Association of Variable Star Observers (www.aavso.org).



Cephied Variables

* During compression the opacity of a star increases, HeII lines begin to saturate until opacity is high and helium begins to ionize to HeIII, heat builds $k \propto \frac{\rho}{T^{3.5}}$
* Heat builds and star's pressure increases driving the layers back out until He begins to recombine dropping opacity, heat, and pressure, star begins to fall back in on itself
* Valve like behaviour, open valve, close valve
* Period-Luminosity relationship

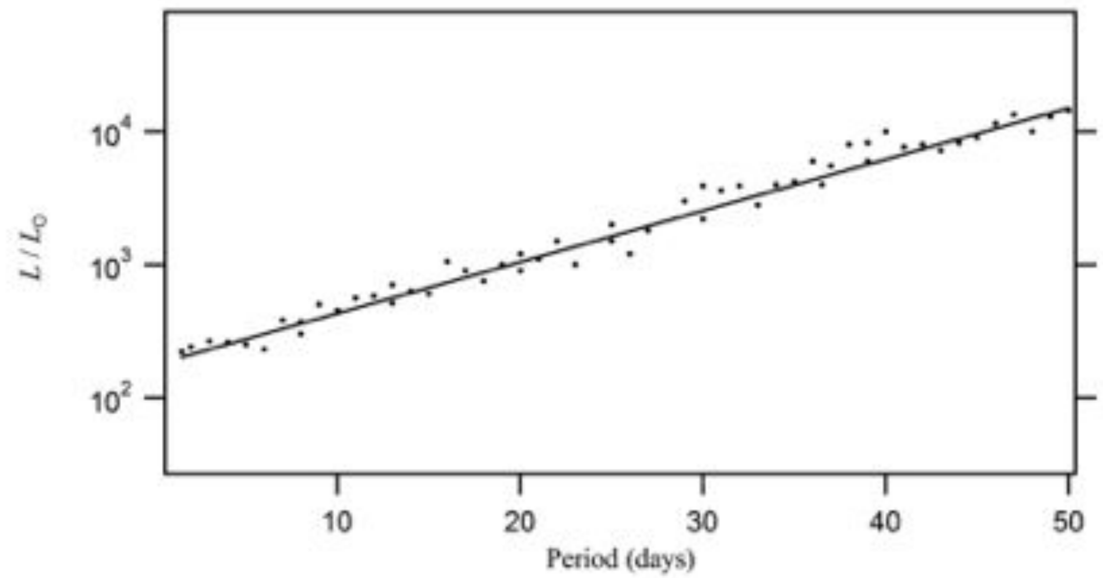


Figure 5.13 Illustration of the period–luminosity relation for classical (or Type-I) Cepheid stars. The dots represent individual classical Cepheid stars while the curve is the best linear fit of the data points giving the relation between the period and the luminosity.

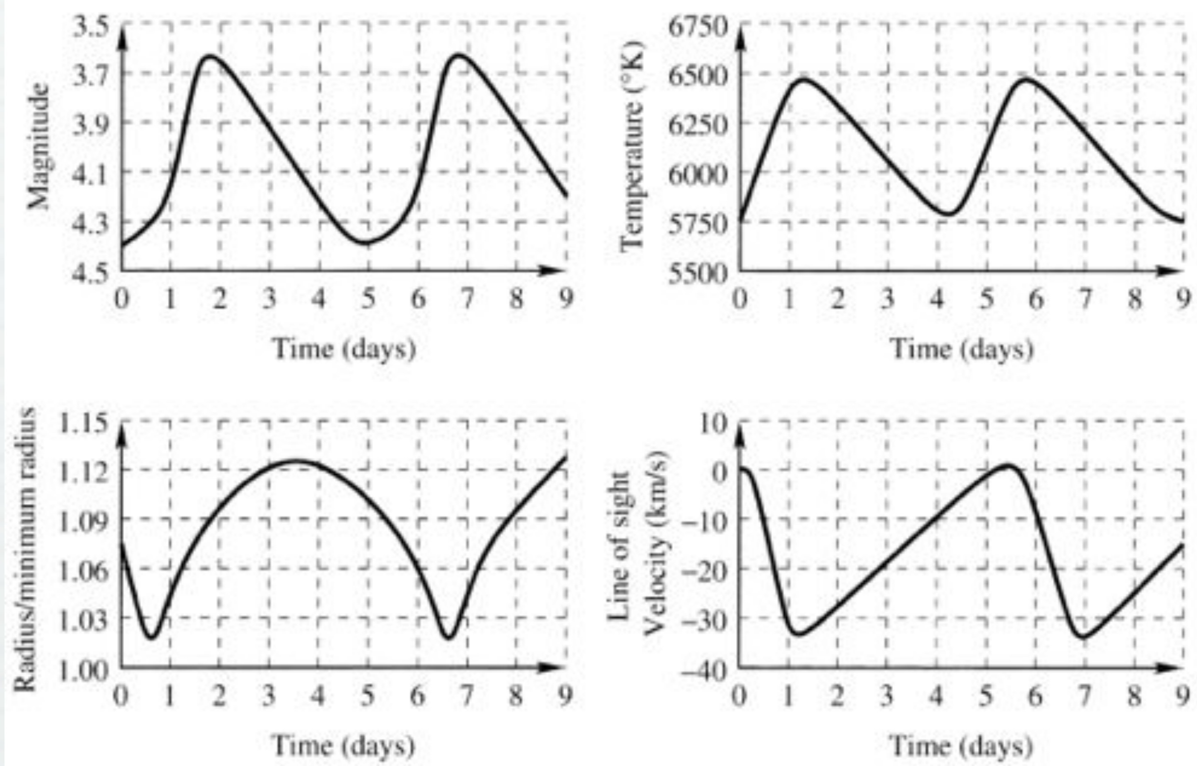


Figure 5.14 Magnitude, temperature, radius and line of sight velocity (or radial velocity) of its surface as a function of time for the classical Cepheid star δ Cephei (which is the prototype for this type of variable stars). The velocity given here is not corrected for the radial velocity of δ Cephei with respect to Earth, which has a value of approximately −16 km/s. Since the maximum of the velocity curve above is approximately found at 0 km/s, the velocities above −16 km/s represent contraction, while those below this value correspond to the phase of expansion. Reproduced with permission from Percy, J.R., Understanding Variable Stars, Cambridge University Press, Cambridge (2007).

Classical cephieds

* Periods days-months
* 4-20 times more massive than sun
* 100,000 brighter than sun
* Classify distances in our local group and determine the Hubble Constant
* Local group - 54 galaxies over 3.1 MPc

Type 2 cepheids

* periods 1-50 days
* old
* half a solar mass
* used to determine distances to globular clusters, distances in the milky way, and distances to local galaxies

Physics of Pulsation

* Not Hydrostatic Equilibrium

$$\rho \frac{d^2 r}{dt^2} = -G \frac{M_r \rho}{r^2} - \frac{dP}{dr}$$

- Consider a central point of mass M equal to the stars mass
- Surround this by a shell of mass m at radius r
- Between the shell and M there is a massless gas of pressure P which supports the shell
- Newton's Law is then

$$m \frac{d^2 R}{dt^2} = -G \frac{Mm}{R^2} + 4\pi R^2 P$$

Linearization

* We take quantities P_0, R_0, etc which do not change and add to the quantities δP and δR which are small and temporally dependent and are called perturbations to the static terms
* Variations of order $(\delta P, R)^2$ and higher are small and may be neglected

Linearization

* In equilibrium $G\frac{Mm}{R_0^2} = 4\pi R_0^2 P_0$
* We let $R = R_0 + \delta R \qquad P = P_0 + \delta P$
* Newton's 2nd law then becomes

$$m\frac{d^2(R_0 + \delta R)}{dt^2} = -\frac{GMm}{(R_0 + \delta R)^2} + 4\pi(R_0 + \delta R)(P_0 + \delta P)$$

Linearization

* Using $\dfrac{1}{(R_0+\delta R)^2} \approx \dfrac{1}{R_0^2}(1-2\dfrac{\delta R}{R_0})$
* Newton's second law becomes

$$m\frac{d^2(\delta R)}{dt^2} = -\frac{GMm}{R_0^2} + \frac{2GMm}{R_0^3}\delta R + 4\pi R_0^2 P_0 + 8\pi R_0 P_0 \delta R + 4\pi R_0^2 \delta P$$

* The second temporal derivative of R0 vanishes and we have dropped terms of order two in the perturbed quantities and higher
* the first and third term cancel

* Leaving us with

$$m\frac{d^2(\delta R)}{dt^2} = \frac{2GMm}{R_0^3}\delta R + 8\pi R_0 P_0 \delta R + 4\pi R_0^2 \delta P$$

* To further simplify matters we relate the perturbed radius and pressure by assuming the oscillations are adiabatic and may be related by

$$PV^\gamma = K \rightarrow P\frac{4}{3}\pi R^3 = K$$

* Which gives us $\frac{\delta P}{P_0} = -3\gamma\frac{\delta R}{R_0}$
* Which allows us to eliminate the perturbed pressure from the linearized equation

* Furthermore $8\pi R_0 P_0 = \frac{2GMm}{R_0^3}$
* Which results in $\frac{d^2(\delta R)}{dt^2} = -(3\gamma - 4)\frac{GM}{R_0^3}\delta R$
* Which is a simple, second order, linear, homogenous differential equation
* For γ > 4/3 this is solved by $\delta R = ASin(\omega t)$
* Or the radius pulsed with a frequency

$$\omega^2 = (3\gamma - 4)\frac{GM}{R_0^3}$$

* Leading to a pulsation period

$$\Omega = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{4}{3}\pi G\rho_0(3\gamma - 4)}}$$

The pulsation period

* Is inversely proportional to density
* Lower density giants have shorter pulsation periods than higher density lower mass stars

Polytropes

* Or how to get a simplified model of a star
* Combine (neglecting radiative acceleration)

$$\frac{dP(r)}{dr} = \rho(r)[g_{rad}(r) - g(r)] = \rho(r)[g_{rad}(r) - \frac{GM(r)}{r^2}]$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

* We manipulate the equation for pressure and take it's radial derivative

$$\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -G\frac{dM_r}{dr}$$

* Upon using the equation for conservation of mass this becomes $\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -4\pi G\rho$

* Which can be written as Poisson's equation for the gravitational potential or $\nabla^2\phi = 4\pi G\rho$

* This equation can not fully describe a star as it neglects the specifics of energy generation and temperature profile but it suffices as an approximation
* We now employ the relationship $P(\rho) = K\rho^{\gamma}$
* and get $\frac{\gamma K}{r^2}\frac{d}{dr}[r^2\rho^{\gamma-2}\frac{d\rho}{dr}] = -4\pi G\rho$
* We rewrite gamma in terms of the polytropic index

$$\gamma = \frac{n+1}{n}$$

* and get $\frac{n+1}{n}\frac{K}{r^2}\frac{d}{dr}[r^2\rho^{(1-n)/n}\frac{d\rho}{dr}] = -4\pi G\rho$

- We recast the density into a dimensionless form based on the central density $\rho(r) = \rho_c[\theta_n(r)]^n$
- where theta is between 0 and 1
- Substituting and simplifying gives us

$$(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta_n}{dr}\right] = -\theta_n^n$$

- Changing variables such that

$$\lambda_n = \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

- and $r = \lambda_n \xi$

* gives us the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}[\xi^2\frac{d\theta_n}{d\xi}] = -\theta_n^n$$

* Solving this equation gives us the density profile and the pressure profile $P_n(r) = K\rho_n^{(n+1)/n}$
* If we assume a constant mean molecular weight we also get the temperature profile
* We still need two constants of integration
* First, assume the pressure goes to zero at the surface and is maximal in the core

* Then $\theta(\xi = 0) = 1$
* and $\frac{d\theta(\xi)}{d\xi}|_{\xi=0} = 0$
* There are just three analytical solutions

$$n = 0 \rightarrow \theta(\xi) = 1 - \frac{\xi^2}{6}$$

$$n = 1 \rightarrow \theta(\xi) = \frac{sin(\xi)}{\xi}$$

$$n = 5 \rightarrow \theta(\xi) = \frac{1}{(1 + \xi^2/3)^{1/2}}$$

*

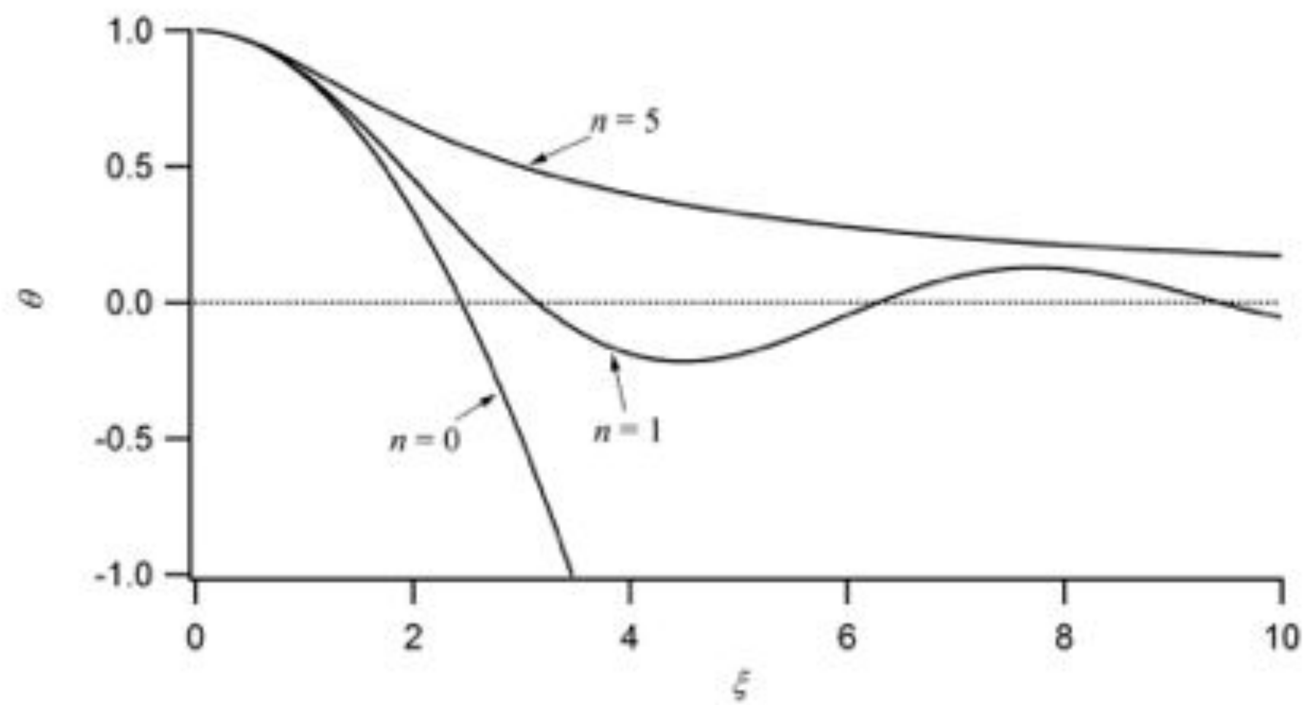


Figure 5.6 The dependence of $\theta(\xi)$ as a function of ξ for polytropic models with various indices.

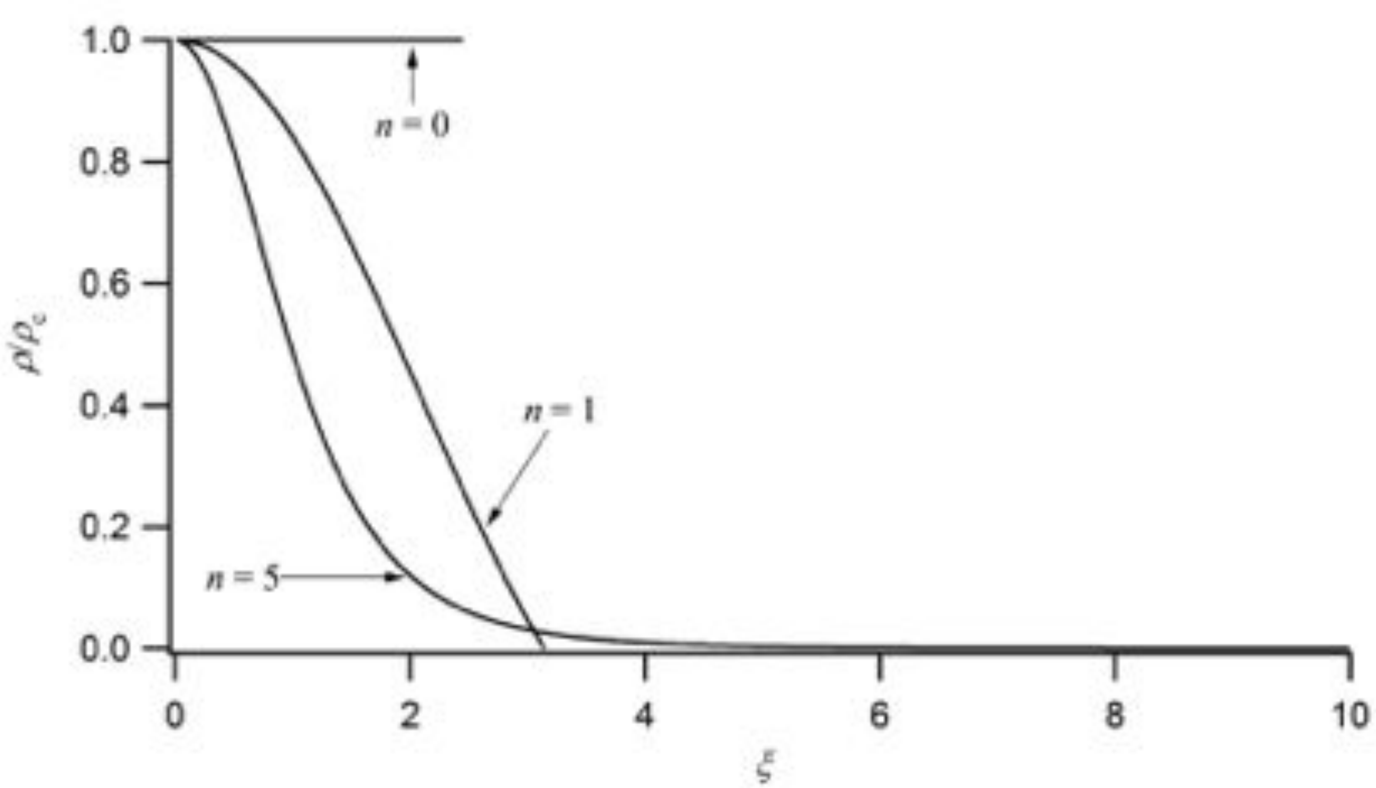


Figure 5.7 The dependence of ρ/ρ_c as a function of ξ for polytropic models of various indices.

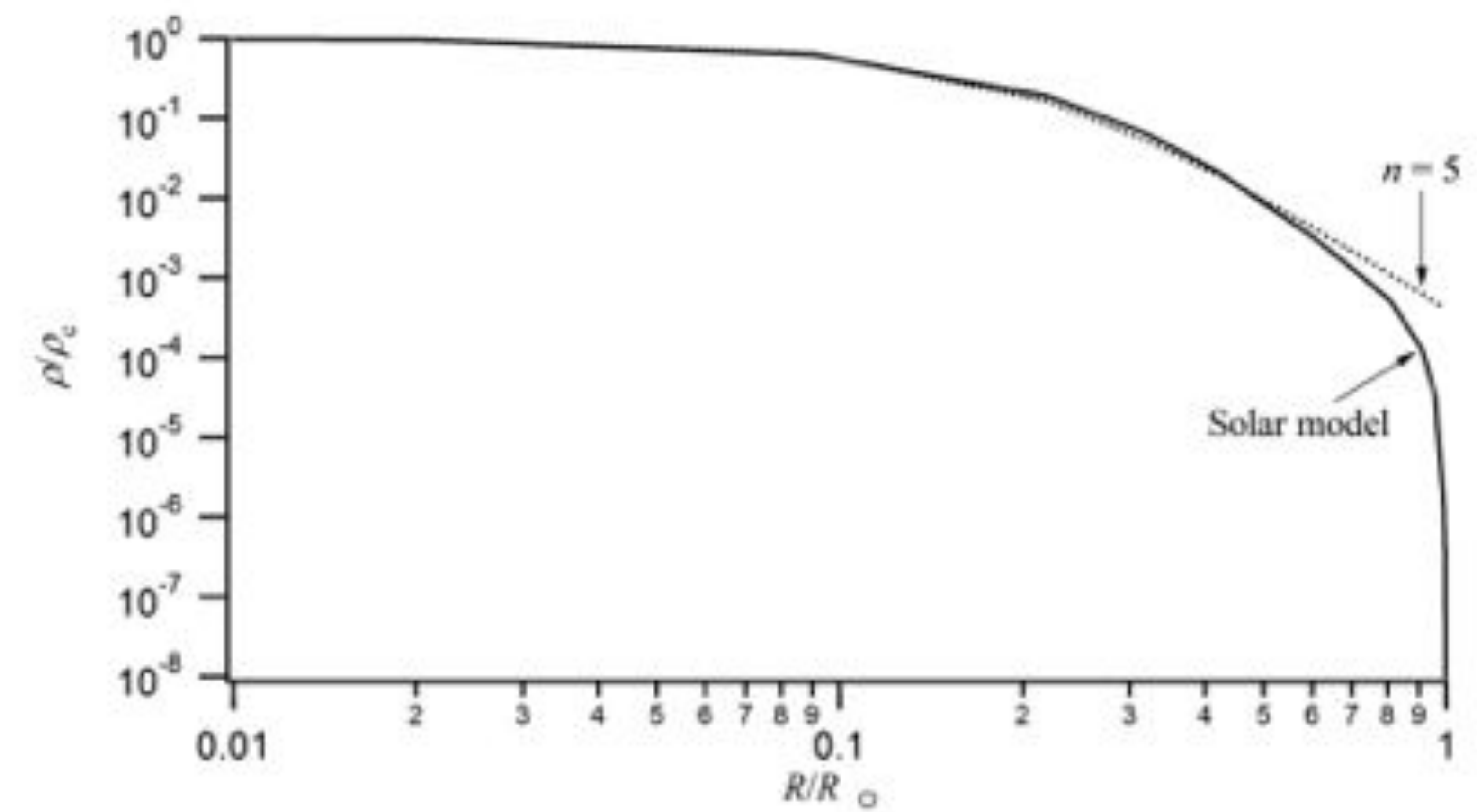


Figure 5.8 The dependence of ρ/ρ_c as a function of radius for a polytropic model with $n = 5$ as compared to a detailed numerical model of the Sun. The data for this theoretical model is found in Table 5.1 (Section 5.5).

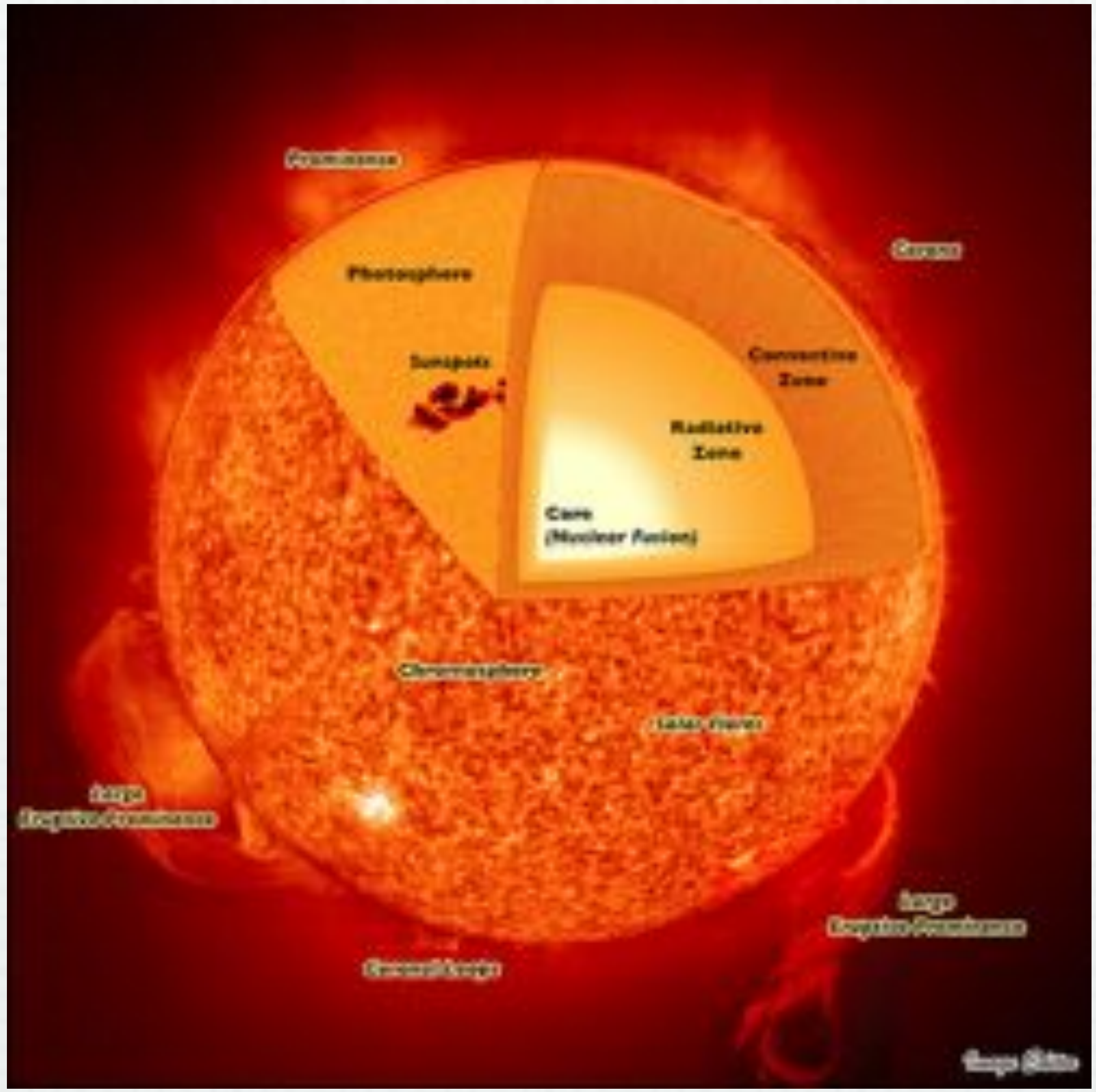
To calculate the mass

$$M_{star} = \int_0^{R_{star}} 4\pi r^2 \rho(r) dr = 4\pi\lambda^3\rho_c \int_0^{\xi_0} \xi^2\theta^n(\xi) d\xi$$

How about the mass enclosed in the core?

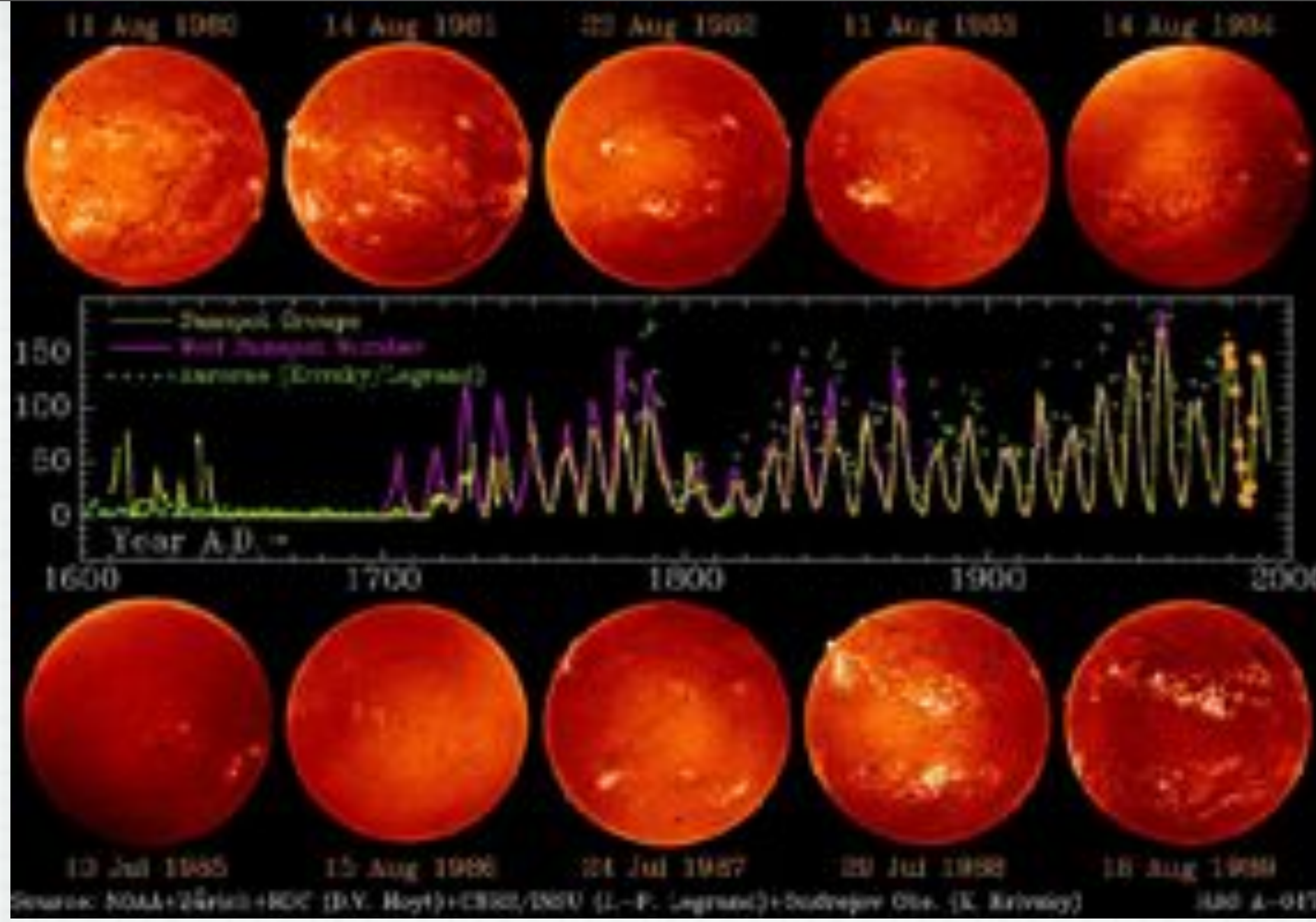
Solar structure

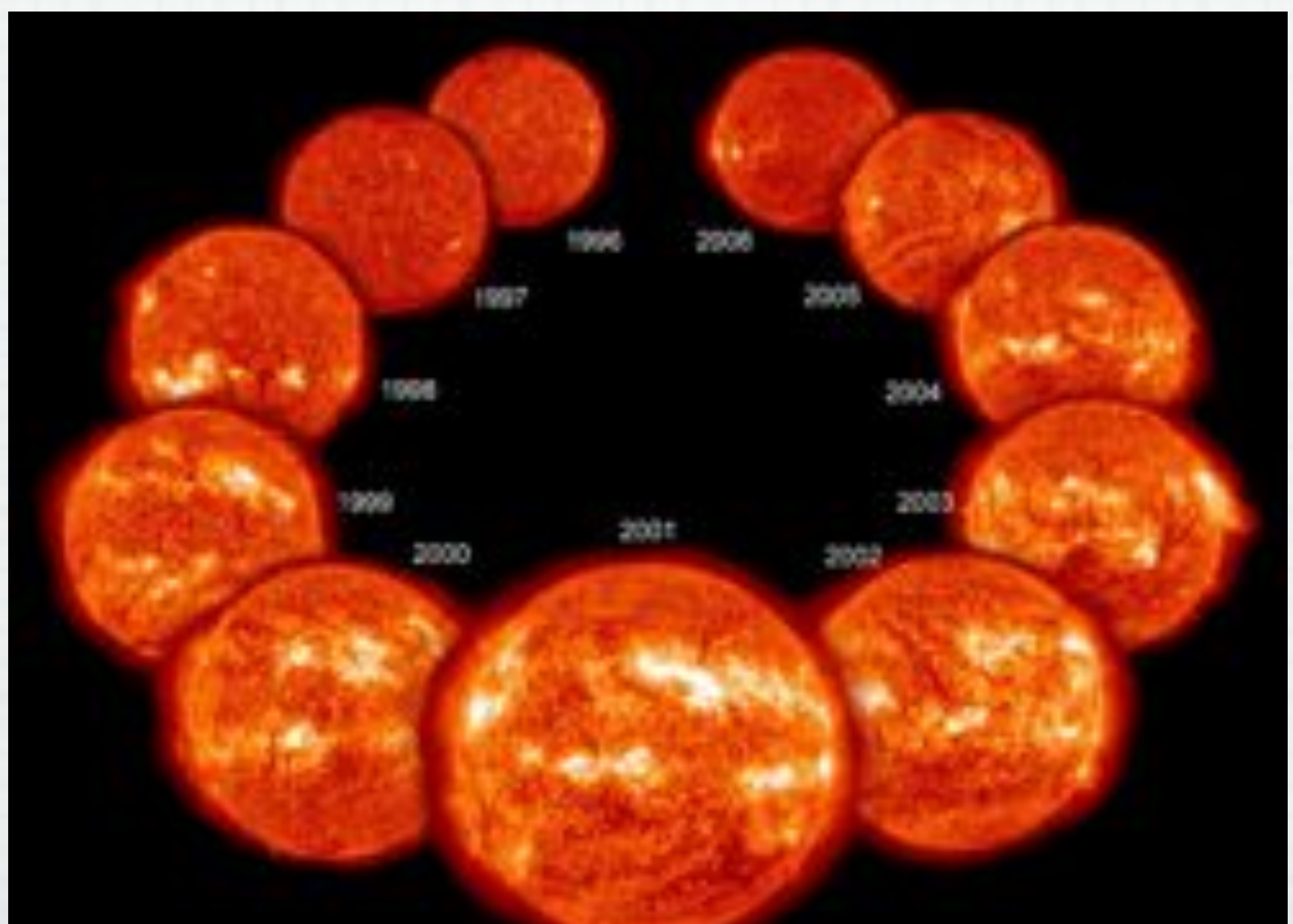
* We'll discuss the structure of our own star



Our sun

* Is a class g2V main seqeunce star comprising 99.86% of the mass in the solar system
* It rotates differentially taking 25 days for one rotation at the equator and 35 days at the poles
* This differential rotation causes the magnetic field lines of the sun to wrap up and generate a magnetic dynamo which generates new magnetic field out of rotational and gravitational energy
* This also allows for the solar cycle and interesting behaviours like flares and CMEs


150
100
50
0
Year A.D.
1600
1700
1800
1900
2000


1996
1997
1998
1999
2000
2001
2002
2003
2004
2005
2006

The core of the sun

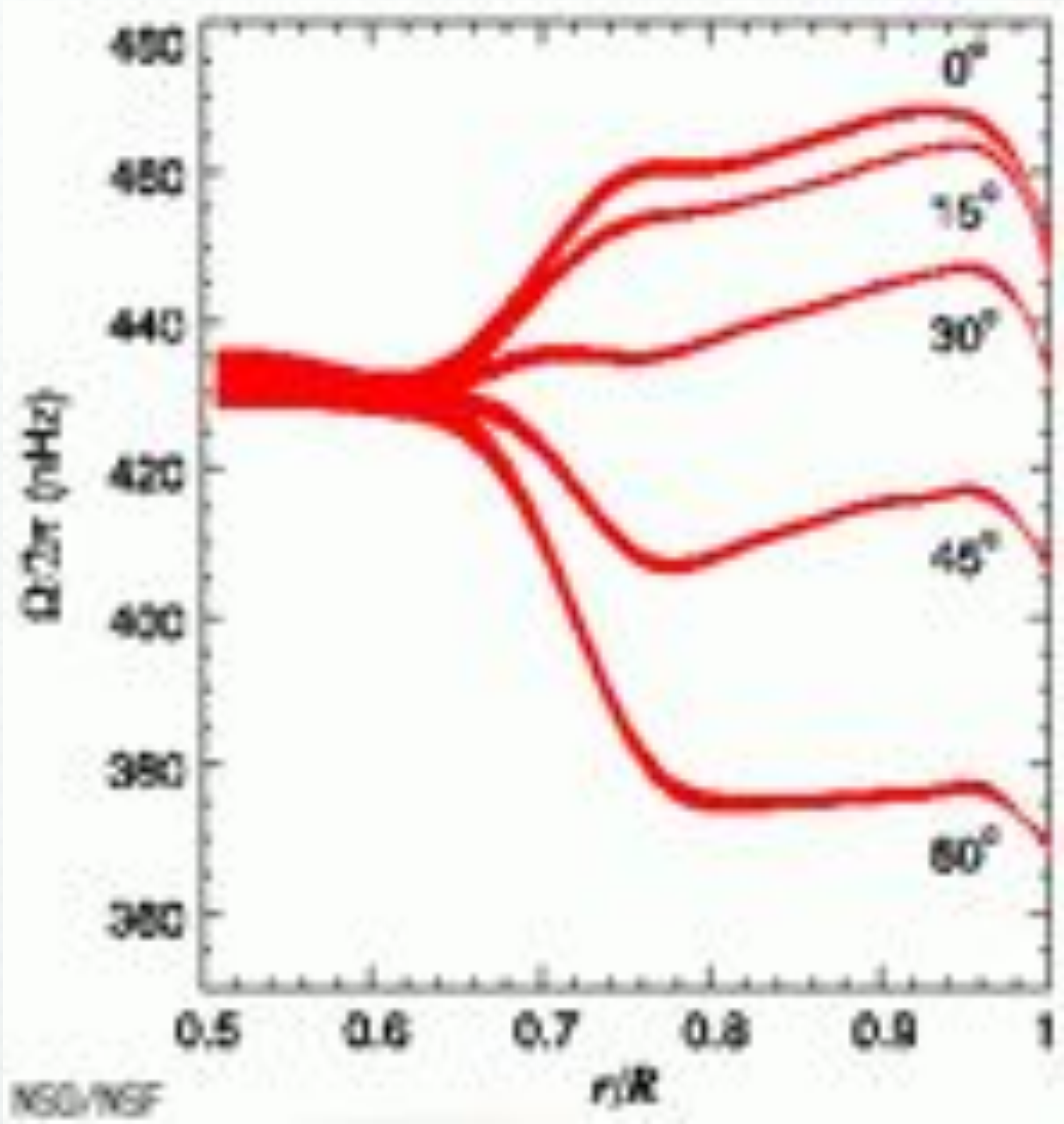
* Has a density of about 150 g/cm^3 or 150 times that of water and extend to about 20-25 percent of the radius of the sun
* Produces energy via the pp chain with about 9.2 x 10^{37} reactions per second converting 6.2 x 10^{11} kg of proton into helium every second while releasing the energy equivalent of 9.192 x 10^{10} megatons of tnt per second
* The gamma ray photons in the sun take 10,000-200,000 years to reach the photosphere
* The temperature in the core is about 16,000,000 K or 30,000,000 F
* Both the temperature and density drop as one moves farther from the center

The radiative zone of the sun

* The radiative zone is where radiation dominates the transport of energy and extends from the core to about 70% the radius of the sun
* No fusion occurs here
* The temperature drops from 7 million kelvin to 2 million kelvin
* The density drops from 20 g/cm^3 to .2g/cm^3
* The radiative zone ends at the tachocline

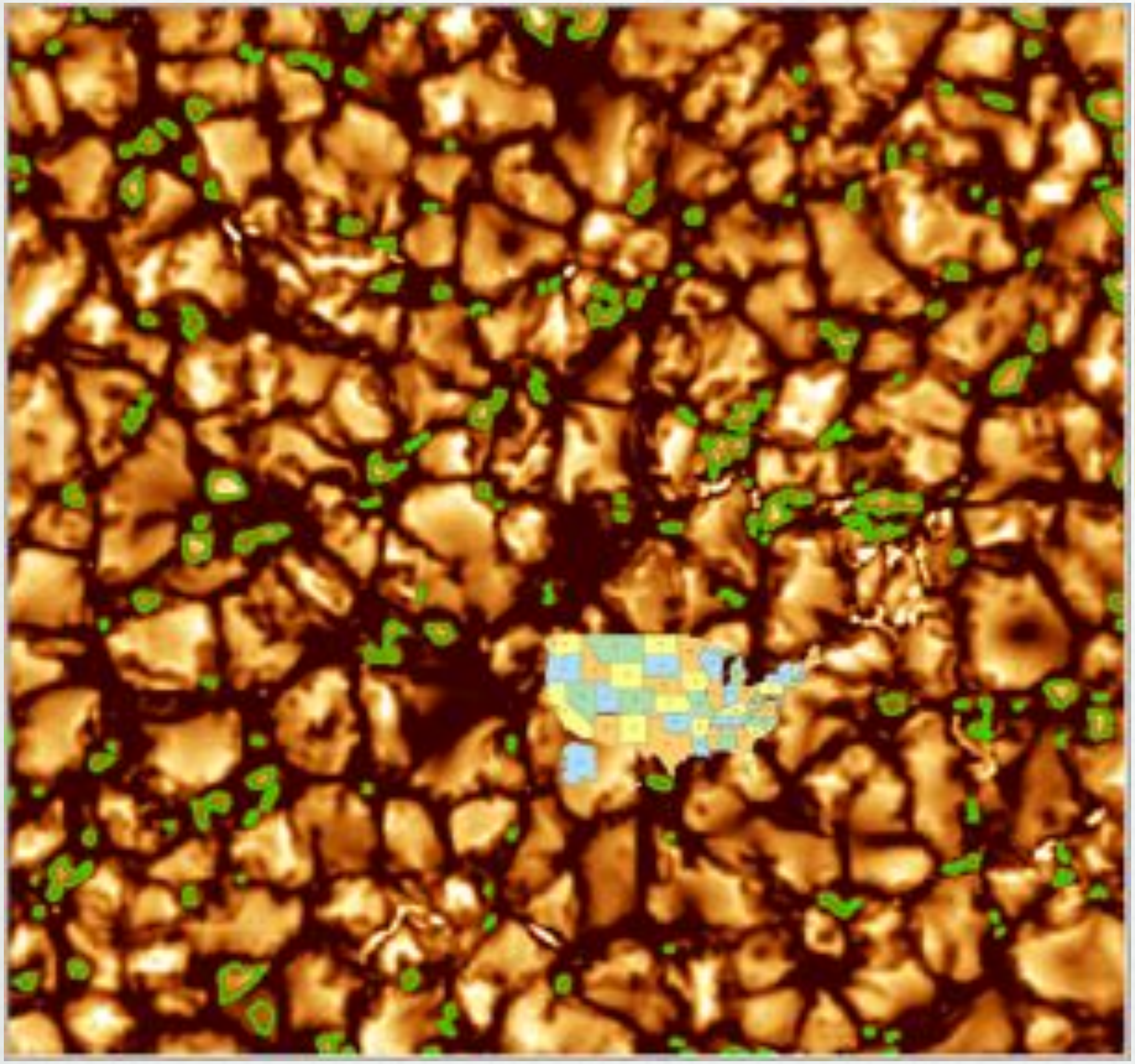
The tachocline

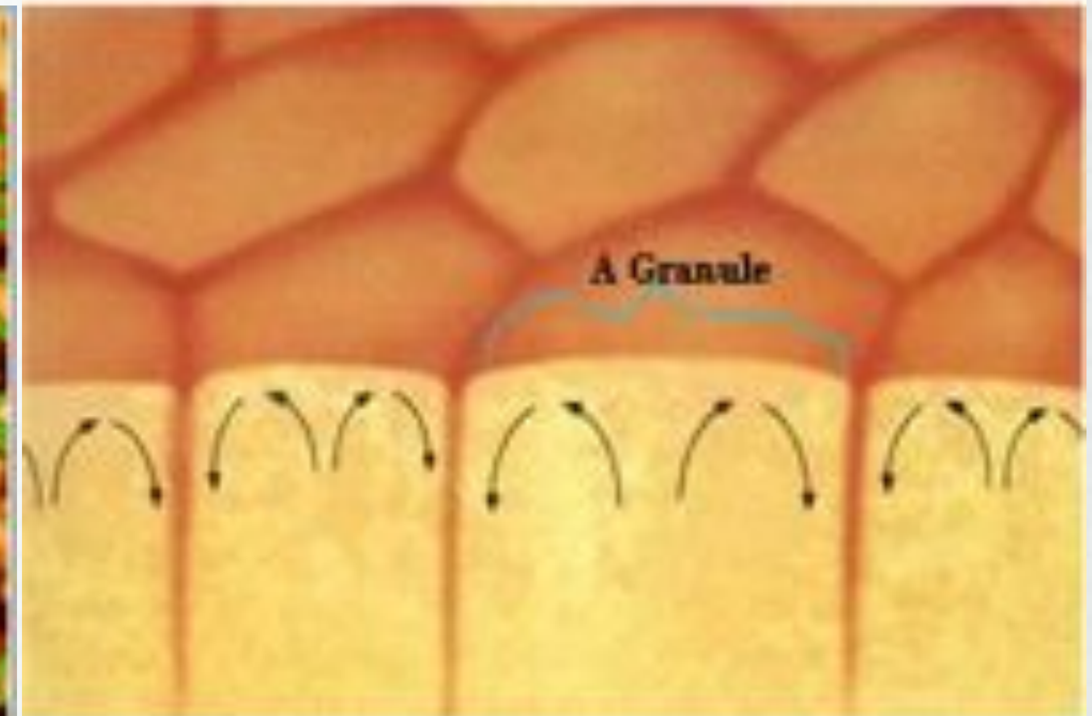
* The transition region between the radiative zone and the convective zone
* The radiative zone rotates as a solid body
* The convective zone and above rotates differentially
* Is approximately .04 solar radius



The convective regions

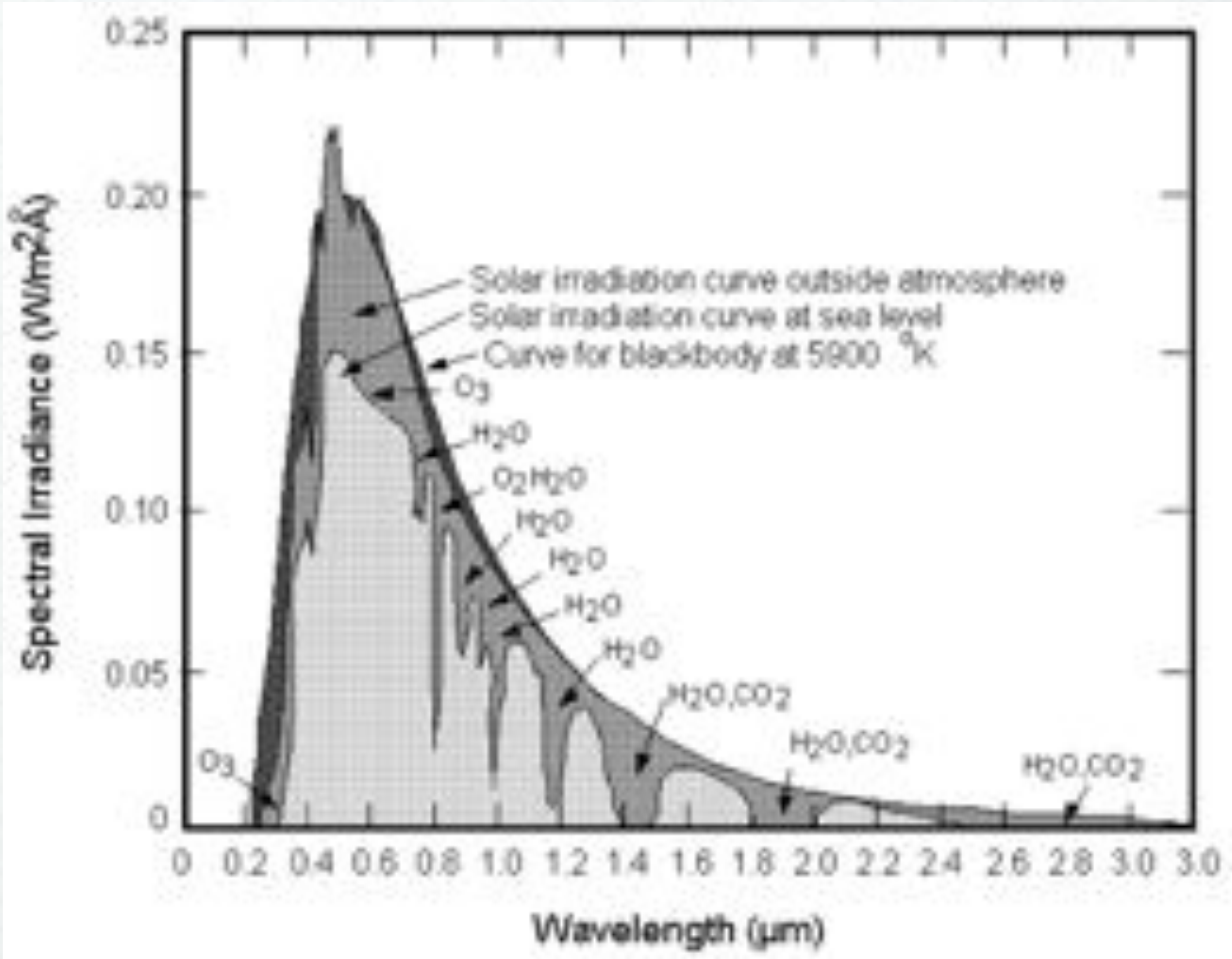
* Has a density of .2 g/cm3 or 1/16000th earths atmosphere at sea level
* The lower temperature results in partial ionization which means the opacity to radiative transport is greater
* Lower density material picks up heat and expands (adiabatic expansion) causing convection to be the primary driver of energy transport
* At the surface of the sun, the photosphere, the material releases heat, cools, and sinks
* Results in granulation and benard cells




A Granule

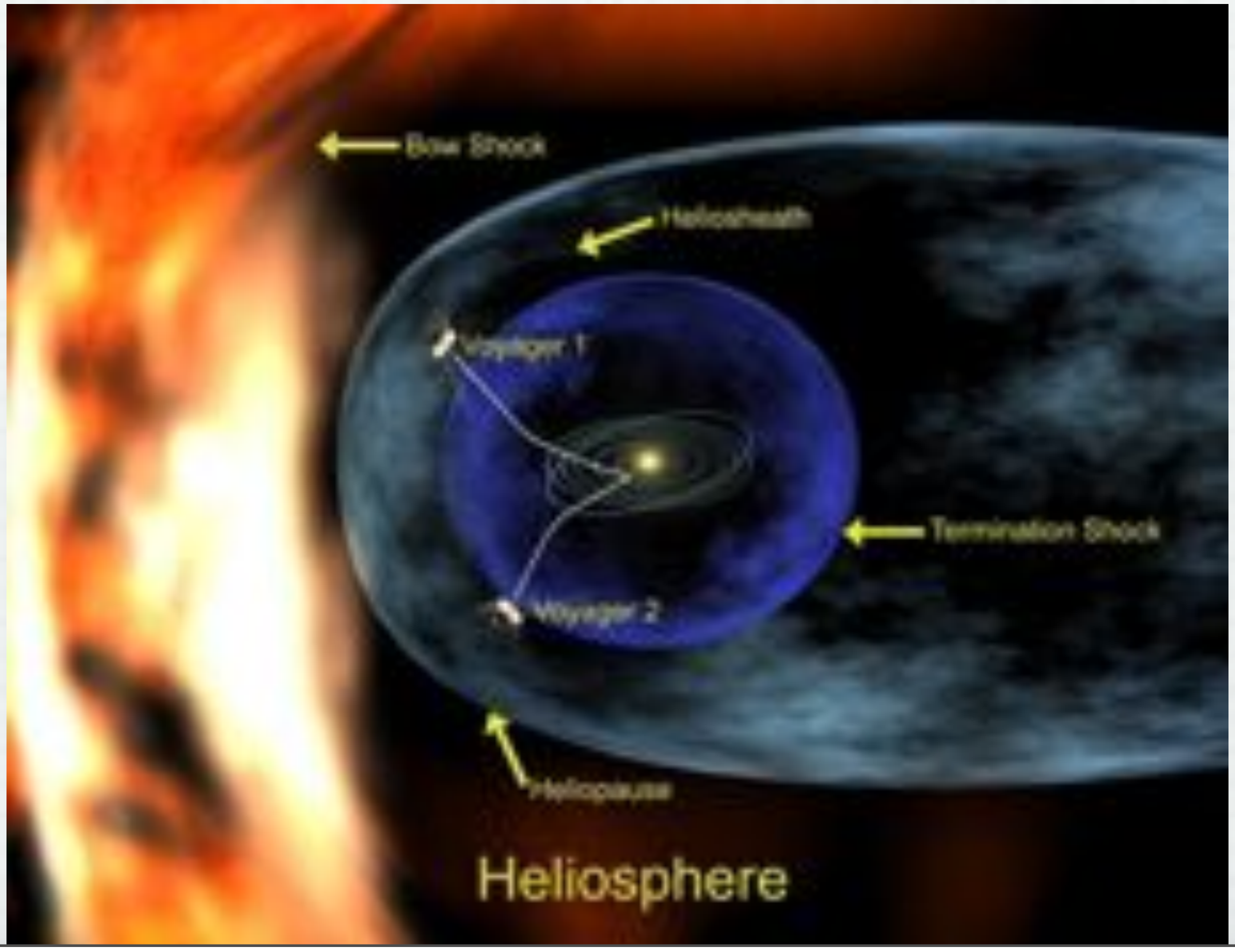
The photosphere

- Or surface of last scattering where photons fly free
- Is about 5800 K
- 3% ionized
- about .37% as dense as our atmosphere at see level
- Emits almost like a blackbody
-



Solar Atmosphere

* Chromosphere
  * Temperature rises to 20,000 K
* Transition Region
  * Temperature rises to 1,000,000 K due to full helium ionization which reduces cooling
* Corona
  * Temperature rises millions of degrees kelvin
  * Solar wind leaves here at one or two million kelvin
  * Temperature probably due to magnetic reconnection
* Heliosphere
  * region carved out by solar wind, ends at 50 AU in interstellar space


Bow Shock
Heliosheath
Voyager 1
Termination Shock
Voyager 2
Heliopause
Heliosphere

Atmosphere observable

* With SDO
* http://sdo.gsfc.nasa.gov/

Ideal Gas

* Provided by the electrons and ions

$$P = n_{tot}k_BT = (n_e + n_{ions})k_BT = \frac{\rho k_B T}{\mu m_H}$$

* where $\frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_{ions}}$
* Evolves over time
* Define the mass fraction $X_i = \frac{\rho_i}{\rho} = \frac{n_i m_i}{\sum n_j m_j}$

* Now the total ion pressure may be written as

$$P_{ions} = (\sum n_j) k_B T$$

* From our equation for mass fraction

$$n_i = \frac{X_i}{m_i} \rho$$

* and the mass of each ion is equal to its nucleon mass (closer enough) $m_i = A_i m_H$

* Leading to

$$n_i = \frac{X_i}{A_i m_H} \rho$$

* or $P_{ions} = (\sum \frac{X_j}{A_j})\frac{\rho k_B T}{m_h} \rightarrow \frac{1}{\mu_{ions}} = \sum \frac{X_j}{A_j}$
* For electrons, assuming complete ionization, where z is the nuclear charge $n_e = \sum z_j n_j$
* Leading to $n_e = \frac{\rho}{m_H} \sum \frac{z_j A_j}{X_j} \rightarrow \frac{1}{\mu_e} = \sum \frac{z_j A_j}{X_j}$
* or $P_e = \frac{\rho k_B T}{m_H} \sum \frac{z_j A_j}{X_j}$ where $\frac{1}{\mu_e} = \frac{n_e}{\rho / m_H}$
* measures the free electrons per nucleon
* Now $\frac{1}{\mu} = \sum \frac{X_j(1 + z_j)}{A_j}$

For a star

* X = hydrogen fraction, Y = helium fraction, Z =1-X-Y = metal fraction
* for complete ionization $\mu = \frac{2}{3X + Y/2 + 1}$
* For completely neutral $\mu = \frac{1}{X + Y/4}$
* As hydrogen goes to helium the mean molecular weight increases therefore pressure decreases, the core contracts, the star heats, slowly, stars heat as they age on the main sequence.

Stars slowly heat up

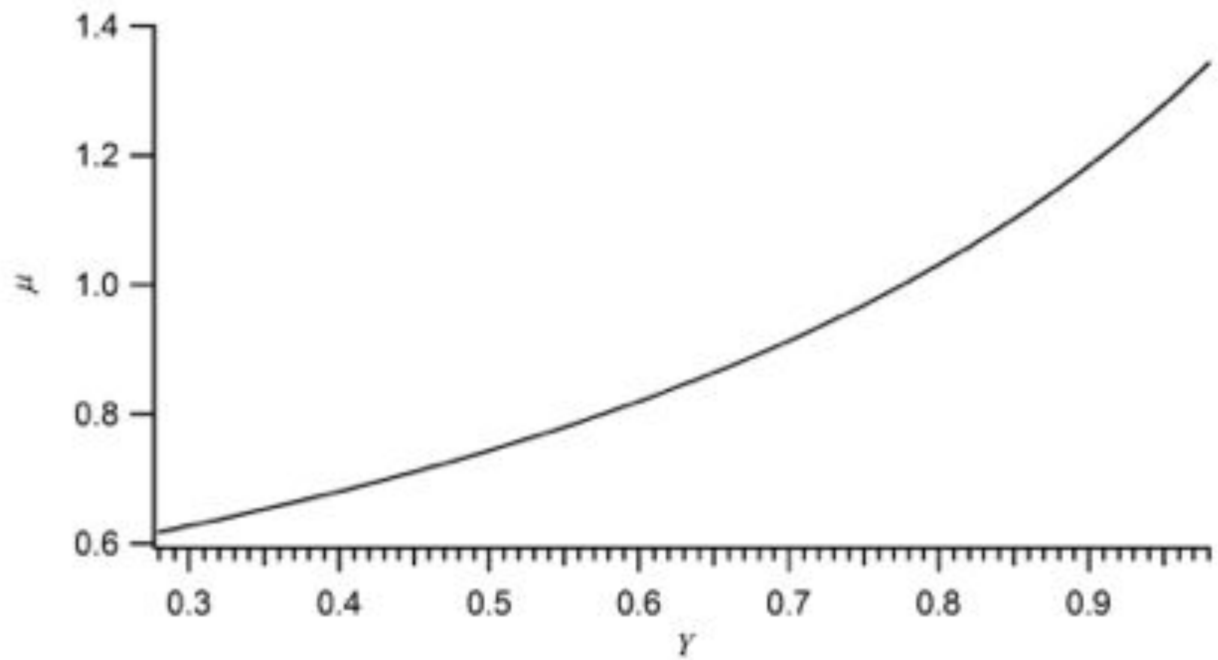


Figure 6.9 The variation of the mean molecular weight in completely ionised plasma as hydrogen is fused and the mass fraction of helium (Y) increases. A metallicity of $Z = 0.02$ was used here.