Phys 396 2026

Lecture 7

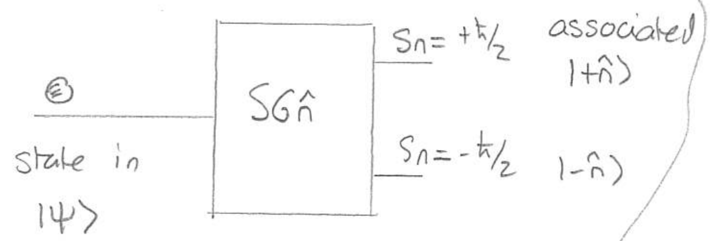
Weds: Read

Fri: HW 3

States and measurement outcomes

The structure of quantum theory for spin-½ particles allows one to determine outcomes of measurements as follows

Suppose that a particle is in the state $|\psi\rangle$ prior to a SG$\hat{n}$ measurement. Then list the possible measurement outcomes and the associated states



outcomes	associated state
$S_n = +\hbar/2$	$\lvert +\hat{n}\rangle$ ← gives outcome with certainty
$S_n = -\hbar/2$	$\lvert -\hat{n}\rangle$

↳ The probabilities with which each outcome can occur are

$$\text{Prob}(S_n = +\hbar/2) = |\langle +\hat{n}|\psi\rangle|^2$$

Bohr rule

$$\text{Prob}(S_n = -\hbar/2) = |\langle -\hat{n}|\psi\rangle|^2$$

↳ Given that one outcome did occur the state immediately after is that which would give the outcome with certainty.

e.g. get $S_n = +\hbar/2 \Rightarrow$ state after is $|+\hat{n}\rangle$

Calculations are facilitated by two requirements. First

The states associated with distinct outcomes of one measurement are orthonormal. So for any direction $\hat{n}$

$$\langle +\hat{n} | +\hat{n} \rangle = 1 \qquad \langle -\hat{n} | +\hat{n} \rangle = 0$$

$$\langle -\hat{n} | -\hat{n} \rangle = 1 \qquad \langle +\hat{n} | -\hat{n} \rangle = 0$$

Second

If $\hat{n}$ is a direction described by spherical angles θ, ϕ

$$|+\hat{n}\rangle = \cos\theta/2 \, |+\hat{z}\rangle + e^{i\phi} \sin\theta/2 \, |-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin\theta/2 \, |+\hat{z}\rangle - e^{i\phi} \cos\theta/2 \, |-\hat{z}\rangle$$

A further mathematical result is that any normalized state

$$|\psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

where a_+, a_- are complex numbers such that $|a_+|^2 + |a_-|^2 = 1$ can be expressed as

$$|\psi\rangle = e^{i\alpha} |+\hat{n}\rangle$$

for some real α and some direction $\hat{n}$. The quantity $e^{i\alpha}$ is called a global phase and does not affect probabilities of measurement outcomes. So

For any state $|\psi\rangle$, the is some direction $\hat{n}$ such that $SG\hat{n}$ gives $S_n = +\hbar/2$ with certainty.

1 Spin state measurement outcome probabilities

a) A particle in state $|+\hat{\boldsymbol{y}}\rangle$ is subjected to a SG $\hat{\boldsymbol{z}}$ measurement. Use the bra and ket formalism to determine the probabilities of the outcomes.

b) A particle in state $|-\hat{\boldsymbol{x}}\rangle$ is subjected to a SG $\hat{\boldsymbol{y}}$ measurement. Use the bra and ket formalism to determine the probabilities of the outcomes.

Answer: a) $\text{Prob}\,(S_z = +\hbar/2) = |\langle +\hat{z}|+\hat{y}\rangle|^2$

$\text{Prob}\,(S_z = -\hbar/2) = |\langle -\hat{z}|+\hat{y}\rangle|^2$

Requires $\quad |+\hat{y}\rangle = \cos\theta/2\,|+\hat{z}\rangle + e^{i\phi}\sin\theta/2\,|-\hat{z}\rangle \quad$ and $\theta = \pi/2 \quad \phi = \pi/2$

$\Rightarrow \quad |+\hat{y}\rangle = \cos\pi/4\,|+\hat{z}\rangle + e^{i\pi/2}\sin\pi/4\,|-\hat{z}\rangle$

$|+\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{i}{\sqrt{2}}|-\hat{z}\rangle$

So $\quad \langle +\hat{z}|+\hat{y}\rangle = \frac{1}{\sqrt{2}}\cancel{\langle +\hat{z}|+\hat{z}\rangle} + \frac{i}{\sqrt{2}}\cancel{\langle +\hat{z}|-\hat{z}\rangle}^{0} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \text{Prob}\,(S_z = +\hbar/2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$\text{Prob}\,(S_z = -\hbar/2) = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$

b) This requires

$\text{Prob}(S_y = +\hbar/2) = |\langle +\hat{y}|+\hat{x}\rangle|^2$

$\text{Prob}\,(S_y = -\hbar/2) = |\langle -\hat{y}|+\hat{x}\rangle|^2$

Now $\quad |+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle$

$\langle +\hat{y}|+\hat{x}\rangle = \left[\frac{1}{\sqrt{2}}\langle +\hat{z}| - \frac{i}{\sqrt{2}}\langle -\hat{z}|\right]\left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle\right]$

$= \frac{1}{2} - \frac{i}{2}$

$$|\langle +\hat{y}|+\hat{x}\rangle|^2 = \left|\frac{1}{2}-\frac{i}{2}\right|^2$$

$$= \left(\frac{1}{2}-\frac{i}{2}\right)^*\left(\frac{1}{2}-\frac{i}{2}\right)$$

$$= \left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right) = \frac{1}{4}+\frac{1}{4}$$

Thus

$$\text{Prob}\,(S_y = +\hbar/2) = \frac{1}{2}$$

similarly $|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$ will give

$$\langle -\hat{y}|+\hat{x}\rangle = \frac{1}{2}+\frac{i}{2}$$

and

$$\text{Prob}\,(S_y = -\hbar/2) = \frac{1}{2}$$

Measurement operators

Can we combine the outcomes and the associated states for a given measurement into a single mathematical quantity so that the quantity can be used to describe outcomes?

Consider a $SG\hat{z}$ measurement. Suppose that prior to the measurement the particle is in state $|\psi\rangle$. Then

$$\text{Prob}\left(S_z = +\hbar/2\right) = |\langle +\hat{z}|\psi\rangle|^2$$

$$= \left(\langle +\hat{z}|\psi\rangle\right)^* \langle +\hat{z}|\psi\rangle$$

since, for any complex number, u, $|u|^2 = u^* u$. Also the inner product satisfies

$$\left(\langle +\hat{z}|\psi\rangle\right)^* = \langle \psi|+\hat{z}\rangle .$$

Thus

$$\text{Prob}\left(S_z = +\hbar/2\right) = \langle \psi|\underbrace{+\hat{z}\rangle\langle +\hat{z}}|\psi\rangle$$

refers to measurement and outcome

We would like to extract the quantity $|+\hat{z}\rangle\langle +\hat{z}|$ that only refers to the measurement ($SG\hat{z}$) and the outcome ($S_z = +\hbar/2$), but not the probability, which requires the state of the particle ($|\psi\rangle$) prior to the measurement.

Can we make a mathematical meaning for $|+\hat{z}\rangle\langle +\hat{z}|$?

Consider what this means in terms of vectors

$$|+\hat{z}\rangle \leadsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle +\hat{z}| \leadsto \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Thus

$$|+\hat{z}\rangle\langle+\hat{z}| \rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix}(1\ 0)$$

This is not an inner product type of multiplication. It does not produce a scalar. Rather it is a multiplication of two matrices and will produce a new matrix.

Note that matrices multiply according to

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

complex numbers $\equiv$ matrix entries

So the product is

$$\hat{A}\hat{B} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

1st row on 1st column — 1st row on 2nd column — 1st row on 3rd column

2nd row 1st column

$$= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + B_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \\ A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} & A_{31}B_{13} + A_{32}B_{23} \end{pmatrix}$$

So multiplying a $\overset{\text{rows}}{l}\times\overset{\text{columns}}{m}$ matrix with a $m \times n$ matrix gives a $l \times n$ matrix.

So out example $\begin{pmatrix}1\\0\end{pmatrix}(1\ 0)$ multiplies a 2×1 matrix with a 1×2 matrix and should produce a 2×2 matrix

2 SG $\hat{z}$ measurement operators

Use the row and column vectors to determine the matrices representing:

a) $|+\hat{z}\rangle\langle+\hat{z}|$.

b) $|-\hat{z}\rangle\langle-\hat{z}|$.

Answer: a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\times 1 & 1\times 0 \\ 0\times 1 & 0\times 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\times 0 & 0\times 1 \\ 1\times 0 & 1\times 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

This gives a more abstract notion. Given any state $|+\hat{n}\rangle$, the quantity $|+\hat{n}\rangle\langle+\hat{n}|$ is an operator that maps kets to kets

$$\begin{array}{c} \text{Ket in} \\ |\psi\rangle \end{array} \longrightarrow \boxed{\begin{array}{c} \text{operator} \\ |+\hat{n}\rangle\langle+\hat{n}| \end{array}} \longrightarrow \begin{array}{c} \text{Ket out} \\ \underbrace{\left(\langle+\hat{n}|\psi\rangle\right)}_{\text{number}} |+\hat{n}\rangle \end{array}$$

The last follows from

$$\begin{aligned} |+\hat{n}\rangle\langle+\hat{n}|\psi\rangle &= |+\hat{n}\rangle \underbrace{\left(\langle+\hat{n}|\psi\rangle\right)}_{\text{number}} \\ &= \left(\langle+\hat{n}|\psi\rangle\right) |+\hat{n}\rangle \end{aligned}$$

In the case of $SG\hat{z}$ measurements we therefore define two <u>measurement operators</u>

$$\boxed{\begin{aligned} \hat{\Pi}_{+z} &:= |+\hat{z}\rangle\langle+\hat{z}| \\ \hat{\Pi}_{-z} &:= |-\hat{z}\rangle\langle-\hat{z}| \end{aligned}}$$

We can use these to determine probabilities of measurement outcomes via

> If a particle is in state $|\psi\rangle$ prior to measurement, then
>
> $$\text{Prob}(S_z = +\hbar/2) = \langle\psi|\hat{\Pi}_{+z}|\psi\rangle$$
>
> $$\text{Prob}(S_z = -\hbar/2) = \langle\psi|\hat{\Pi}_{-z}|\psi\rangle$$

3 Measurement operators and probabilities of outcomes

Suppose that a spin-1/2 particle is in state

$$|\psi\rangle = a_+ \,|+\hat{\mathbf{z}}\rangle + a_- \,|-\hat{\mathbf{z}}\rangle$$

where a_+ and a_- are complex numbers. Use the measurement operators $\hat{\Pi}_{+z}$ and $\hat{\Pi}_{-z}$ to determine the probabilities of the outcomes of the measurements.

Answer: $\text{Prob}(S_z = +\hbar/2) = \langle\psi|\hat{\Pi}_{+z}|\psi\rangle$

and $|\psi\rangle = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$

$\Rightarrow \langle\psi| = |\psi\rangle^\dagger = (a_+^* \;\; a_-^*)$

So

$$\langle\psi|\hat{\Pi}_{+z}|\psi\rangle = (a_+^* \; a_-^*)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = (a_+^* \; a_-^*)\begin{pmatrix} a_+ \\ 0 \end{pmatrix}$$

$$= |a_+|^2$$

$\Rightarrow \text{Prob}(S_z = +\hbar/2) = |a_+|^2$

Then $\langle\psi|\hat{\Pi}_{-z}|\psi\rangle = (a_+^* \; a_-^*)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_+ \\ a_- \end{pmatrix}$

$$= (a_+^* \; a_-^*)\begin{pmatrix} 0 \\ a_- \end{pmatrix} = |a_-|^2$$

$\Rightarrow \text{Prob}(S_z = -\hbar/2) = |a_-|^2$