Phys 396 F2026

Lecture 6

Fri: HW 2; 5pm

Mon: Read My notes 3.3
2.1.3

Spin-$\frac{1}{2}$ quantum states

A spin-$\frac{1}{2}$ quantum system is one which is such that when any single component of spin is measured, the outcome is one of two possibilities: $+\hbar/2$ or $-\hbar/2$.

Observations then allow us to define states for these particles:

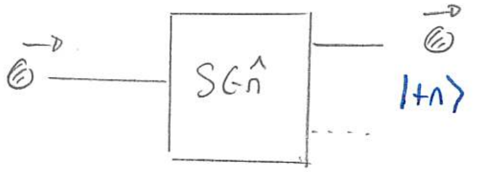
Given any direction, described by unit vector $\hat{n}$ there are two special states for a spin-$\frac{1}{2}$ particle with meanings:

$|+\hat{n}\rangle \Rightarrow$ measure S_n outcome is $S_n = +\hbar/2$ with certainty

$|-\hat{n}\rangle \Rightarrow$ measure S_n outcome is $S_n = -\hbar/2$ with certainty

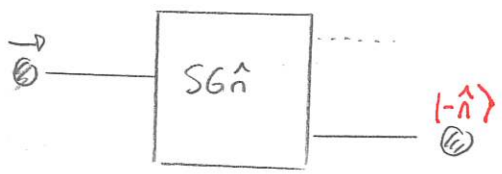
We require that measurements be repeatable and this implies knowledge about states emerging after measurement

Suppose an SG$\hat{n}$ measures S_n. Then the two possibilities are



outcome is $S_n = +\hbar/2$

state after is $|+\hat{n}\rangle$



outcome is $S_n = -\hbar/2$

state after is $|-\hat{n}\rangle$

In general, a spin-½ particle will not yield one measurement outcome with certainty, regardless of how well its state is prepared. The statistics of the measurement outcomes are described via:

Suppose that $\hat{m}$ is a unit vector describing the state of the particle and $\hat{n}$ is a unit vector describing the component to be measured. If the state of the particle prior to measurement is $|+\hat{m}\rangle$ then the measurement of S_n yields:

outcome	probability	state after
$S_n = +\hbar/2$	$(1+\hat{m}\cdot\hat{n})/2$	$|+\hat{m}\rangle$
$S_n = -\hbar/2$	$(1-\hat{m}\cdot\hat{n})/2$	$|-\hat{m}\rangle$

In this context there are infinitely many states for a spin-½ particle:

$$\{\, |+\hat{n}\rangle \mid \text{all unit vector directions } \hat{n} \,\}$$

We need to develop the mathematics associated with these. The tasks will be

Associate states with vectors

$$|+\hat{n}\rangle \overset{??}{=} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \vdots \end{pmatrix}$$

→

Relate different states e.g.

$|+\hat{x}\rangle \overset{??}{=}$ combination of $|+\hat{z}\rangle, |-\hat{z}\rangle$

→

Determine probabilities of measurement outcomes

Mathematics of spin-½ states

As before the kets will be elements of a complex vector space, with the operations previously encountered. So

- * For any ket $|+\hat{n}\rangle$ we can form a bra $\langle +\hat{n}| = |+\hat{n}\rangle^{\dagger}$
- * For any ket and any bra we can form the inner product e.g. $\langle +\hat{m}|+\hat{n}\rangle$

We will then require

A ket that describes the state of a spin-½ particle must be normalized. So for any $\hat{n}$

$$\langle +\hat{n}|+\hat{n}\rangle = 1$$

We will also require:

Consider an SG$\hat{n}$ measurement. The two states associated with the two distinct outcomes (i.e. the state that gives $S_n = +\hbar/2$ with certainty and the state that gives $S_n = -\hbar/2$ with certainty) are orthogonal and form a basis for the space of all kets.

For example $|+\hat{z}\rangle$ and $|-\hat{z}\rangle$ are associated with the outcomes of SG$\hat{z}$ and they form a basis. So any state can be expressed as

$$|+\hat{n}\rangle = a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle$$

where a_+ and a_- are complex numbers. Since these two kets form a basis the vectors are two-dimensional.

We conventionally represent these using

$$|+\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\hat{z}\rangle \rightsquigarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are orthonormal. So

$$\langle +\hat{z} | +\hat{z} \rangle = 1 \qquad \langle +\hat{z} | -\hat{z} \rangle = 0$$

$$\langle -\hat{z} | -\hat{z} \rangle = 1 \qquad \langle -\hat{z} | +\hat{z} \rangle = 0$$

Probabilities of measurement outcomes

The bra/ket formalism is used to determine probabilities of measurement outcomes according to the Bohr rule.

Suppose that an SG$\hat{n}$ measurement (for specified direction $\hat{n}$) is performed on a particle initially in state $|\psi\rangle$. Then the probabilities of the measurement outcomes are

$$\text{Prob}\left(S_n = +\hbar/2\right) = |\langle +\hat{n} | \psi \rangle|^2$$

$$\text{Prob}\left(S_n = -\hbar/2\right) = |\langle -\hat{n} | \psi \rangle|^2$$

where $|+\hat{n}\rangle$ and $|-\hat{n}\rangle$ are the states associated with the measurement (i.e. yield each of the outcomes with certainty)

Suppose that the particle is in state

$$|\psi\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle \quad \rightsquigarrow \quad a_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

where a_+, a_- are complex

Then we require that the state is normalized.

$$\langle\psi| = \left(a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle\right)^\dagger$$

$$= a_+^* \langle +\hat{z}| + a_-^* \langle -\hat{z}|$$

and

$$\langle\psi|\psi\rangle = |a_+|^2 + |a_-|^2 = 1$$

Then,

$$\text{Prob}\,(S_z = +\hbar/2) = |\langle +\hat{z}|\psi\rangle|^2$$

so

$$\langle +\hat{z}|\psi\rangle = \langle +\hat{z}|\left[a_+|+\hat{z}\rangle + a_-|-\hat{z}\rangle\right]$$

$$= a_+ \overset{1}{\cancel{\langle +\hat{z}|+\hat{z}\rangle}} + a_- \overset{0}{\cancel{\langle +\hat{z}|-\hat{z}\rangle}}$$

$$= a_+$$

$$\text{Prob}\,(S_z = +\hbar/2) = |a_+|^2$$

Similarly

$$\text{Prob}\,(S_z = -\hbar/2) = |a_-|^2$$

1 Measurement outcome probabilities

Consider the state

$$|\Psi\rangle = 3i\,|+\hat{z}\rangle + 4\,|-\hat{z}\rangle\,.$$

a) Normalize this state and denote the result by $|\psi\rangle$.

b) Assume that a particle in this state is subjected to a SG $\hat{z}$ measurement. Determine the probabilities of the two outcomes.

c) ~~Consider the state $|\phi\rangle := e^{i\varphi}\,|\psi\rangle$ where φ is any real number. Express this in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$. Assume that a particle in this state is subjected to a SG $\hat{z}$ measurement. Determine the probabilities of the two outcomes.~~

a) $\langle\Psi|\Psi\rangle = \left[(3i)^*\langle+\hat{z}| + 4\langle-\hat{z}|\right]\left[3i\,|+\hat{z}\rangle + 4\,|-\hat{z}\rangle\right]$

$= (-3i)(3i) + 4\cdot4$

$= 25$

we can thus normalize via $1/\sqrt{25}$. So

$|\psi\rangle = \frac{1}{\sqrt{5}}\,3i\,|+\hat{z}\rangle + \frac{1}{\sqrt{5}}\,4\,|-\hat{z}\rangle$

b) $\text{Prob}\,(S_z = +\hbar/2) = |\langle+\hat{z}|\psi\rangle|^2$ and $\langle+\hat{z}|\psi\rangle = \langle+\hat{z}|\left[\frac{3i}{\sqrt{5}}\,|+\hat{z}\rangle + \frac{4}{\sqrt{5}}\,|-\hat{z}\rangle\right]$

$= \frac{3i}{\sqrt{5}}\,\overset{1}{\cancel{\langle+\hat{z}|+\hat{z}\rangle}} + \frac{4}{\sqrt{5}}\,\overset{0}{\cancel{\langle+\hat{z}|-\hat{z}\rangle}}$

$= \frac{3i}{\sqrt{5}}$

so $\text{Prob}\,(S_z = +\hbar/2) = \left|\frac{3i}{\sqrt{5}}\right|^2 = \frac{3i}{\sqrt{5}}\left(\frac{3i}{\sqrt{5}}\right)^* = \frac{9}{25}$

Similarly

$\text{Prob}\,(S_z = -\hbar/2) = \left|\frac{4}{\sqrt{5}}\right|^2 = \frac{16}{25}$

Relationship between spin-$1/2$ states

Suppose that the direction $+\hat{n}$ is known. In general

$$|+\hat{n}\rangle = a_+ |+\hat{z}\rangle + a_- |-\hat{z}\rangle$$

Can we determine a_+, a_- knowning $\hat{n}$? This has to be based on experimental observations. The result is:

Given $\hat{n}$ first find the corresponding spherical angle co-ordinates θ, ϕ i.e.

$$\hat{n} = \cos\phi \sin\theta\, \hat{x} + \sin\phi \sin\theta\, \hat{y} + \cos\theta\, \hat{z}$$

then

$$|+\hat{n}\rangle = \cos\theta/2\, |+\hat{z}\rangle + e^{i\phi} \sin\theta/2\, |-\hat{z}\rangle$$

$$|-\hat{n}\rangle = \sin\theta/2\, |+\hat{z}\rangle - e^{i\phi} \cos\theta/2\, |-\hat{z}\rangle$$

See pg 33 - 34 previous notes

2 States in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

a) Express $|+\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.
b) Express $|-\hat{x}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.
c) Express $|+\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.
d) Express $|-\hat{y}\rangle$ in terms of $\{|+\hat{z}\rangle, |-\hat{z}\rangle\}$.

a) for $\hat{x}$ $\quad \theta = \pi/2 \quad \phi = 0$

$$|+\hat{x}\rangle = \cos \pi/4 \, |+\hat{z}\rangle + e^{i0} \sin \pi/4 \, |-\hat{z}\rangle$$

$$= \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle \qquad \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b)

$$|-\hat{x}\rangle = \sin \pi/4 \, |+\hat{z}\rangle - e^{i0} \cos \pi/4 \, |-\hat{z}\rangle$$

$$= \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{1}{\sqrt{2}} |-\hat{z}\rangle \qquad \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

c) for $\hat{y}$ $\quad \theta = \pi/2 \quad \phi = \pi/2$

$$|+\hat{y}\rangle = \cos \pi/4 \, |+\hat{z}\rangle + e^{i\pi/2} \sin \pi/4 \, |-\hat{z}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{i}{\sqrt{2}} |-\hat{z}\rangle$$

$$|-\hat{y}\rangle = \sin \pi/4 \, |+\hat{z}\rangle - e^{i\pi/2} \cos \pi/4 \, |-\hat{z}\rangle = \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{i}{\sqrt{2}} |-\hat{z}\rangle$$

So

$$\boxed{\begin{aligned} |+\hat{x}\rangle &= \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{1}{\sqrt{2}} |-\hat{z}\rangle \\ |-\hat{x}\rangle &= \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{1}{\sqrt{2}} |-\hat{z}\rangle \end{aligned}} \qquad \boxed{\begin{aligned} |+\hat{y}\rangle &= \frac{1}{\sqrt{2}} |+\hat{z}\rangle + \frac{i}{\sqrt{2}} |-\hat{z}\rangle \\ |-\hat{y}\rangle &= \frac{1}{\sqrt{2}} |+\hat{z}\rangle - \frac{i}{\sqrt{2}} |-\hat{z}\rangle \end{aligned}}$$

3 Spin state measurement outcome probabilities

a) A particle in state $|+\hat{x}\rangle$ is subjected to a SG $\hat{z}$ measurement. Use the bra and ket formalism to determine the probabilities of the outcomes.

b) A particle in state $|-\hat{y}\rangle$ is subjected to a SG $\hat{x}$ measurement. Use the bra and ket formalism to determine the probabilities of the outcomes.

a) state in $|+\hat{x}\rangle$.

$$\text{Prob}\left(S_z=+\hbar/2\right) = |\langle +\hat{z}|+\hat{x}\rangle|^2$$

$$\langle +\hat{z}|+\hat{x}\rangle = \langle +\hat{z}|\left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle\right] = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\text{Prob}\left(S_z=-\hbar/2\right) = |\langle -\hat{z}|+\hat{x}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$\boxed{\begin{array}{l}\text{Prob}(S_z=+\hbar/2) = \frac{1}{2} \\ \text{Prob}(S_z=-\hbar/2) = \frac{1}{2}\end{array}}$$

b) $$\text{Prob}\left(S_x=+\hbar/2\right) = |\langle +\hat{x}|-\hat{y}\rangle|^2$$

$$|-\hat{y}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle$$

$$|+\hat{x}\rangle = \frac{1}{\sqrt{2}}|+\hat{z}\rangle + \frac{1}{\sqrt{2}}|-\hat{z}\rangle \Rightarrow \langle +\hat{x}| = \frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{1}{\sqrt{2}}\langle -\hat{z}|$$

$$\langle +\hat{x}|-\hat{y}\rangle = \left[\frac{1}{\sqrt{2}}\langle +\hat{z}| + \frac{1}{\sqrt{2}}\langle -\hat{z}|\right]\left[\frac{1}{\sqrt{2}}|+\hat{z}\rangle - \frac{i}{\sqrt{2}}|-\hat{z}\rangle\right] = \frac{1}{\sqrt{2}}\frac{-i}{\sqrt{2}}$$

$$|\langle +\hat{x}|-\hat{y}\rangle|^2 = \left|\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} \Rightarrow \boxed{\text{Prob}\left(S_x=+\hbar/2\right) = \frac{1}{2}}$$

Similarly $\text{Prob}\left(S_x=-\hbar/2\right) = \frac{1}{2}$