Phys 396 S 2026

Lecture 5

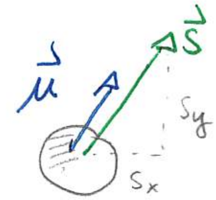
Fri: HW 2 by 5pm (Corrections!)
Read My notes pgs 17–31
Thurs: Seminar, Wubben 264

Quantum theory of spin-½ particles

A spin-½ system is a type of atomic or subatomic system which responds to external magnetic fields (in ways to be described). The basic physics of these is

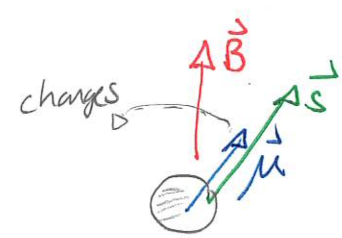
1) spin refers to the angular momentum of the particle. In classical physics this is represented by a vector $\vec{S} = S_x \hat{x} + S_y \hat{y} + S_z \hat{z}$

2) for classical rotating charged particles, the spin is proportional to the magnetic dipole moment $\vec{\mu}$. This describes how the particle responds to magnetic fields



Situating the spin-½ particle in an external magnetic field affects the state of the particle:

1) it can change the magnetic dipole moment and thus an external magnetic field can change the spin state of the particle



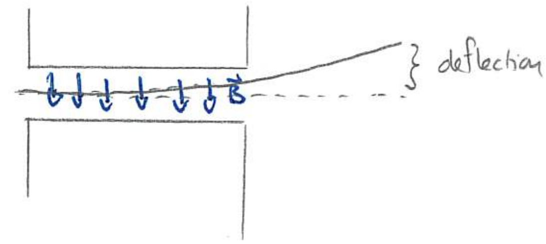
2) it can cause the particle to move in ways that depend on its spin state. This can be harnessed to measure the spin state

These extend to particles which obey quantum theory. We will find that the classical description of spin and measurements of spin needs to be modified.

Stern-Gerlach Experiment

The Stern-Gerlach experiment was originally done in the 1920s and it measures a component of the particle spin by firing the particle into an inhomogeneous magnetic field. The classical analysis predicts:

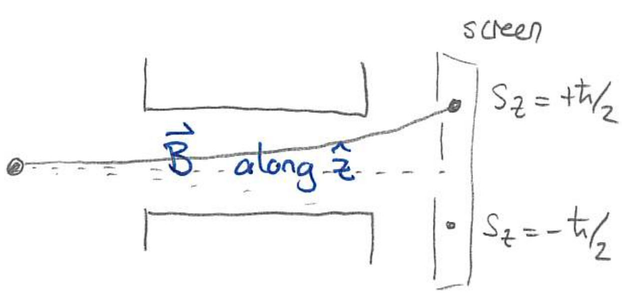
$$\boxed{\text{The deflection of the particle is proportional to the component of the spin along the magnetic field}}$$



DEMO: Toutestquantique video

However when the quantum experiment is done, there are only two deflections possible. So

$$\boxed{\text{Measuring one component of spin can yield one of two possible outcomes: } +\hbar/2 \text{ or } -\hbar/2}$$

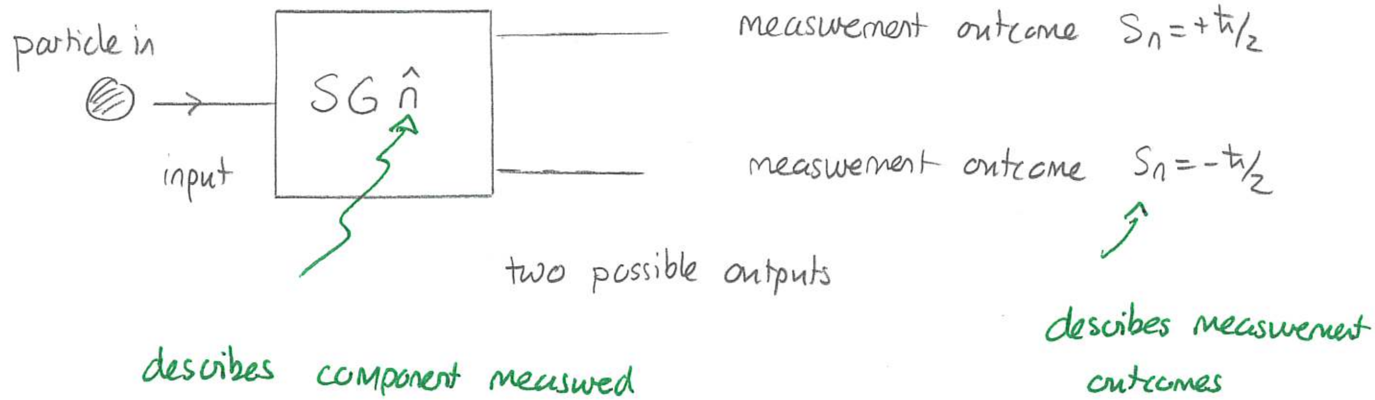


Measures S_z

We could measure any component of spin and the result would be the same

$$\boxed{\begin{array}{l}\text{Suppose that one measures the component of spin along the direction } \hat{n} \text{ (unit vector). Then the outcome will be \underline{one} of two possibilities} \\ \qquad S_n = +\hbar/2 \quad \text{or} \quad S_n = -\hbar/2\end{array}}$$

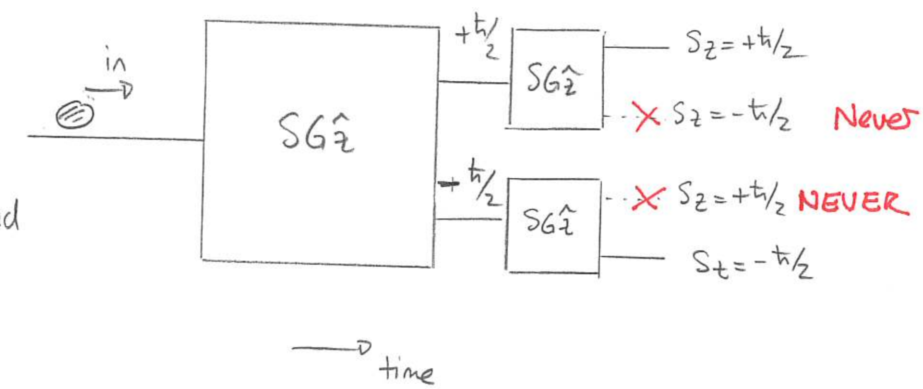
We can represent this schematically as



Successive Stern-Gerlach measurements

The particle that emerges from one Stern-Gerlach measurement can be subjected to a subsequent Stern-Gerlach measurement. Consider the case where successive SG measurements are of the same type. For example two SG$\hat{z}$ measurements.

Evidence indicates that if the first gives $S_z = +\hbar/2$ then the second will give $S_z = +\hbar/2$ with certainty and never $S_z = -\hbar/2$



Similarly if the first gives $S_z = -\hbar/2$ the second will give $S_z = -\hbar/2$ with certainty.

It follows that we can predict with certainty what will happen after the first SG$\hat{z}$ if a second SG$\hat{z}$ follows immediately. Therefore we can describe the state of the particle after. The language is

$|+\hat{z}\rangle \Rightarrow$ If one does a SG$\hat{z}$ measurement the outcome will be $+\hbar/2$ with certainty

$|-\hat{z}\rangle \Rightarrow$ If one does a SG$\hat{z}$ " " " will be $-\hbar/2$ with certainty

We can generalize this to any component and this gives states:

$\vert +\hat{n}\rangle$	$\Rightarrow$	For direction $\hat{n}$, measuring S_n gives $S_n = +\hbar/2$ with certainty ($\vec{S}\cdot\hat{n}$ gives $+\hbar/2$ with certainty)
$\vert -\hat{n}\rangle$	$\Rightarrow$	For direction, $\hat{n}$, measuring S_n gives $S_n = -\hbar/2$ with certainty

The kets that describe the state have form

$$\left| \begin{array}{l}\text{label that describes which} \\ \text{component and the outcome}\end{array} \right\rangle$$

We will often generically write the kets as

$$|\psi\rangle$$

where ψ is a label that describes the state in terms of an outcome of some particular measurement.

1 Single Stern-Gerlach experiments

A stream of particles are all prepared in an identical fashion. These are incident upon a SG $\hat{y}$ apparatus. About 60% of these yield $S_y = +\hbar/2$ and about 40% of these yield $S_y = -\hbar/2$.

a) Based on this could one say that the state of every particle after it has been prepared and before it enters the SG apparatus is $|+\hat{y}\rangle$?

b) Could one say that the state of every particle after it has been prepared and before it enters the SG apparatus is $|-\hat{y}\rangle$?

c) Consider any particle that yields $S_y = +\hbar/2$. After this emerges from the apparatus, can one say what its state is? Explain your answer.

d) Consider any particle that yields $S_y = -\hbar/2$. After this emerges from the apparatus, can one say what its state is? Explain your answer.

e) Describe how one might take a particle in some unknown state and use a SG apparatus to prepare the particle in state $|-\hat{x}\rangle$. When will the procedure definitely fail?

Answer: a) If it were $|+\hat{y}\rangle$ then we would get $S_y = +\hbar/2$ 100%. We did not, so not $|+\hat{y}\rangle$

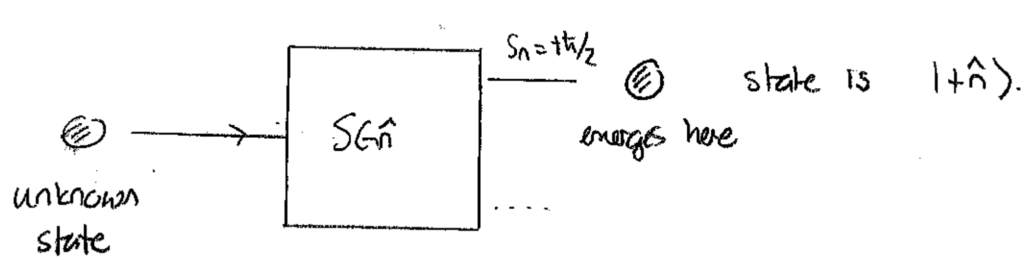
b) If it were $|-\hat{y}\rangle$ then we would get $S_y = -\hbar/2$ 100%. Not $|-\hat{y}\rangle$

c) It will definitely give $S_y = +\hbar/2$ for a subsequent $SG\hat{y}$. So it is in state $|+\hat{y}\rangle$

d) Similar to c) $|-\hat{y}\rangle$

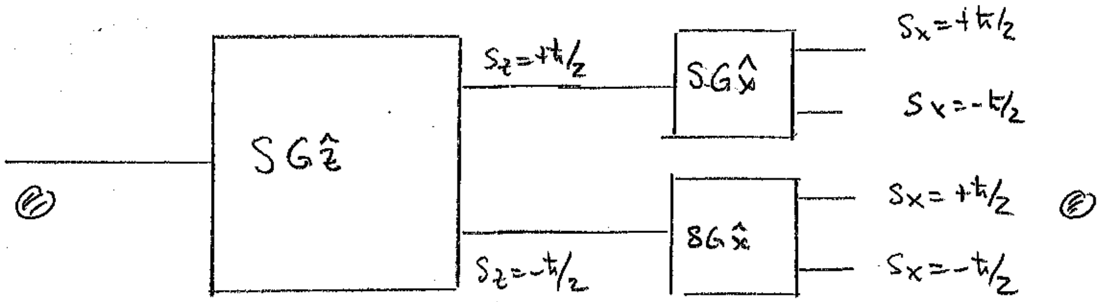
e) Measure S_x using $SG\hat{x}$. If we get $S_x = -\hbar/2$ then state is $|-\hat{x}\rangle$. It can fail if we get $S_x = +\hbar/2$

So we can determine or even prepare a state using $SG\hat{n}$ measurements

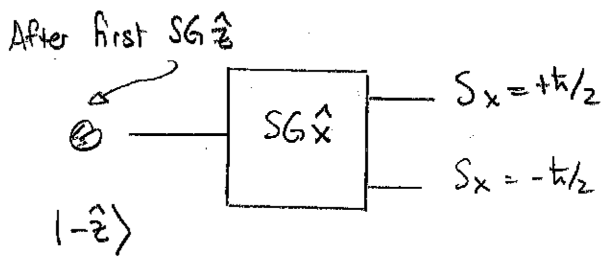


General repeated measurements

We can measure different components successively. Consider first measuring S_z and then S_x. Observations would indicate that all four possible sequences are possible and the particle will only emerge in one of the outputs



Any single particle input will only emerge in one of the four outputs and thus yield a pair of outcomes. The illustration shows $S_z=-\hbar/2$, $S_x=+\hbar/2$. However, another particle prepared in the identical fashion will not necessarily emerge in the same output as before. There is no way to overcome this statistical situation. Simplifying the situation we would find

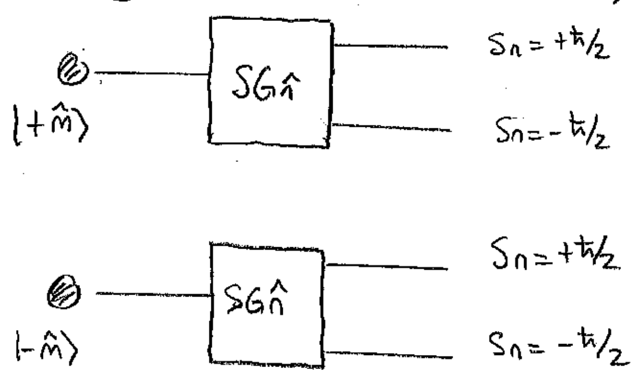


$S_x=+\hbar/2$ with probability $1/2$

$S_x=-\hbar/2$ " " $1/2$

In general we find

For any unit vector directions $\hat{m}$, $\hat{n}$



$$\text{Prob}(S_n=+\hbar/2)=\frac{1+\hat{m}\cdot\hat{n}}{2}$$

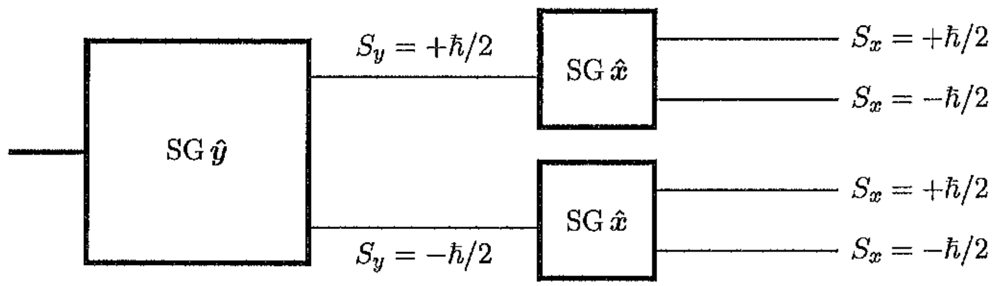
$$\text{Prob}(S_n=-\hbar/2)=\frac{1-\hat{m}\cdot\hat{n}}{2}$$

$$\text{Prob}(S_n=+\hbar/2)=\frac{1-\hat{m}\cdot\hat{n}}{2}$$

$$\text{Prob}(S_n=-\hbar/2)=\frac{1+\hat{m}\cdot\hat{n}}{2}$$

2 Repeated Stern-Gerlach measurements

Two successive SG measurements are oriented as illustrated.

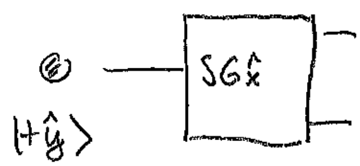


a) Suppose that the particle emerges from the first with $S_y = +\hbar/2$. Determine the probabilities of the two outcomes of the SG $\hat{x}$ measurement.

b) Suppose that the particle emerges from the first with $S_y = -\hbar/2$. Determine the probabilities of the two outcomes of the SG $\hat{x}$ measurement.

c) Suppose that the state of the particle prior to the first is $|+\hat{\mathbf{n}}\rangle$ where $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. Determine the probabilities with which it emerges in all four outputs.

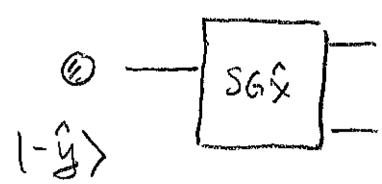
Answer: a) After the first state is $|+\hat{y}\rangle$ and we have



$$\text{Prob}(S_x = +\hbar/2) = \frac{1 + \hat{y}\cdot\hat{x}}{2} = \frac{1}{2}$$

$$\text{Prob}(S_x = -\hbar/2) = \frac{1 - \hat{y}\cdot\hat{x}}{2} = \frac{1}{2}$$

b) After the first the state is $|-\hat{y}\rangle$



$$\text{Prob}(S_x = +\hbar/2) = \frac{1 - \hat{y}\cdot\hat{x}}{2} = \frac{1}{2}$$

$$\text{Prob}(S_x = -\hbar/2) = \frac{1 + \hat{y}\cdot\hat{x}}{2} = \frac{1}{2}$$

c) After the first we have

Outcome	State after measurement of S_y	Prob
$S_y = +\hbar/2$	$\lvert +\hat{y}\rangle$	$\frac{1+\hat{y}\cdot\hat{n}}{2} = \frac{1+1/\sqrt{2}}{2}$
$S_y = -\hbar/2$	$\lvert -\hat{y}\rangle$	$\frac{1-\hat{y}\cdot\hat{n}}{2} = \frac{1-1/\sqrt{2}}{2}$

For both we have

Outcome 1st	Outcome 2nd	
$S_y = +\hbar/2$	$S_x = +\hbar/2$	$\text{Prob}(S_y = +\hbar/2)\,\text{Prob}(S_x = +\hbar/2) = \frac{1}{4}\left(1+\frac{1}{\sqrt{2}}\right)$
$S_y = +\hbar/2$	$S_x = -\hbar/2$	$\frac{1}{4}\left(1+\frac{1}{\sqrt{2}}\right)$
$S_y = -\hbar/2$	$S_x = +\hbar/2$	$\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)$
$S_y = -\hbar/2$	$S_x = -\hbar/2$	$\frac{1}{4}\left(1-\frac{1}{\sqrt{2}}\right)$