Phys 396 F 2026

Lecture 4

Mon: HW1 5pm

Fri: HW 2 5pm

Read: My notes pgs 7–20

Ket vectors

In quantum theory states will be represented by vectors – kets of the form $|v\rangle$ where v is some label. The simplest systems are such that the vectors are two-dimensional. So

$$|v\rangle = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

We can express such kets in terms of a basis $\{|e_1\rangle, |e_2\rangle\}$ where

$$|e_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |e_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So that

$$|v\rangle = v_0|e_0\rangle + v_1|e_1\rangle$$

We will modify the notation a little by relabeling the basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We could regard the $|0\rangle$ basis vector as somehow representing a bit value "0" and the $|1\rangle$ basis vector as representing a bit value "1". Thus we could express any ket as

$$|v\rangle = v_0|0\rangle + v_1|1\rangle \rightsquigarrow v_0\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

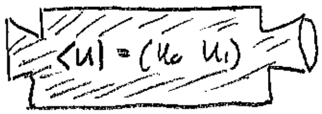
where v_0 and v_1 are complex components. The basis $\{|0\rangle, |1\rangle\}$ forms a "computational basis"

Bra vectors

A bra vector is a row vector

$$\langle u| = (u_0 \;\; u_1)$$

and bra vectors act on kets to produce numbers

Input ket	Bra machine	Output number
$\|v\rangle = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \longrightarrow$		$\longrightarrow \langle u\|v\rangle = (u_0 \; u_1)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = u_0 v_0 + u_1 v_1$

Transformation from a ket to a bra

Given any ket $|v\rangle = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ we can construct an associated bra as $\langle v| = (v_0^* \;\; v_1^*)$ where the asterisk shows complex conjugation. Then the dagger operation $\dagger$ means complex conjugation and transpose. So

$$(v_0^* \;\; v_1^*) = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}^\dagger$$

and thus we write $\langle v| = |v\rangle^\dagger$. So

$$\boxed{\begin{array}{l} \text{The complex conjugate transpose } \dagger \text{ maps a ket to a bra according to} \\ \qquad |v\rangle = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \rightsquigarrow |v\rangle^\dagger = \langle v| = (v_0^* \;\; v_1^*) \\ \text{It also maps a bra to a ket according to:} \\ \qquad \langle u| = (u_0 \;\; u_1) \rightsquigarrow \langle u|^\dagger = |u\rangle = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}^* \end{array}}$$

We can see immediately that

$$\boxed{\text{For any ket } |v\rangle, \quad |v\rangle^{\dagger\dagger} = |v\rangle}$$

The complex conjugate transpose satisfies

$$\boxed{\begin{array}{l}\text{For any } |v\rangle, |w\rangle \text{ and } \alpha, \beta \\ \quad (\alpha|v\rangle + \beta|w\rangle)^\dagger = \alpha^* \langle v| + \beta^* \langle w|\end{array}}$$

Note the special cases:

$$|0\rangle^\dagger = \langle 0| \Rightarrow \begin{pmatrix}1\\0\end{pmatrix}^\dagger = (1\ 0) \Rightarrow \langle 0| = (1\ 0)$$

$$|1\rangle^\dagger = \langle 1| \Rightarrow \begin{pmatrix}0\\1\end{pmatrix}^\dagger = (0\ 1) \Rightarrow \langle 1| = (0\ 1).$$

Inner product

Given two kets $|u\rangle, |v\rangle$ we can define the inner product (generalizes the dot product) as

The inner product of $|u\rangle, |v\rangle$ is $\langle u|v\rangle$

Note that the inner product of $|u\rangle$ with $\alpha|v\rangle + \beta|w\rangle$ is

$$\langle u|(\alpha|v\rangle + \beta|w\rangle) = \alpha\langle u|v\rangle + \beta\langle u|w\rangle.$$

Similarly

$$(\alpha\langle u| + \beta\langle v|)|w\rangle = \alpha\langle u|w\rangle + \beta\langle v|w\rangle$$

Special case involve the computational basis, which is orthogonal:

$$\boxed{\begin{array}{ll}\langle 0|0\rangle = 1 & \langle 0|1\rangle = 0 \\ \langle 1|1\rangle = 1 & \langle 1|0\rangle = 0\end{array}}$$

Then

A ket $|v\rangle$ is normalized if $\langle v|v\rangle = 1$

Furthermore the innerproduct satisfies

$$\langle u|v\rangle = (\langle v|u\rangle)^*$$

1 Ket algebra

Let

$$|u\rangle = |0\rangle - 2i\,|1\rangle \quad \text{and}$$
$$|v\rangle = i\,|0\rangle + |1\rangle$$

a) Check whether each ket is normalized. If not normalize the kets by finding numbers A and B such that $A\,|u\rangle$ and $B\,|v\rangle$ are normalized.

b) Determine $\langle u|v\rangle$ starting with each ket represented as a column vector.

c) Determine $\langle u|v\rangle$ by first constructing $\langle u|$, then expressing this as a combination of $\langle 0|$ and $\langle 1|$ and finally acting with this combination on $|v\rangle$.

Answer: a) $|u\rangle \rightsquigarrow \begin{pmatrix} 1 \\ -2i \end{pmatrix}$ $\quad \langle u|u\rangle = \begin{pmatrix} 1^* & (-2i)^* \end{pmatrix} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2i \end{pmatrix} \begin{pmatrix} 1 \\ -2i \end{pmatrix} \Rightarrow \langle u|u\rangle = 5$$

To normalize let $|w\rangle = A|u\rangle \rightsquigarrow \begin{matrix} A \\ -2iA \end{matrix}$

$$\langle w|w\rangle = \begin{pmatrix} A^* & 2iA^* \end{pmatrix} \begin{pmatrix} A \\ -2iA \end{pmatrix} = A^*A(1+4) = 5(A^*A) = 1$$

Choose $A = \frac{1}{\sqrt{5}}$ then $\frac{1}{\sqrt{5}}|0\rangle - \frac{2i}{\sqrt{5}}|1\rangle$ is normalized

$|v\rangle \rightsquigarrow \begin{pmatrix} i \\ 1 \end{pmatrix}$ $\quad \langle v|v\rangle = \begin{pmatrix} -i & 1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 2$

Let $|w\rangle = B|v\rangle \rightsquigarrow \begin{pmatrix} iB \\ B \end{pmatrix}$. $\quad \langle w|w\rangle = \begin{pmatrix} -iB^* & B^* \end{pmatrix} \begin{pmatrix} iB \\ B \end{pmatrix} = 2B^*B = 1$

Choose $B = \frac{1}{\sqrt{2}}$

$$|v\rangle = \frac{1}{\sqrt{2}} i|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

is normalized.

b) $\langle u| = |u\rangle^{\dagger} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}^{\dagger} = \begin{pmatrix} 1 & (-2i)^{*} \end{pmatrix}$

$\langle u| = \begin{pmatrix} 1 & 2i \end{pmatrix}$

$\langle u|v\rangle = \begin{pmatrix} 1 & 2i \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = 3i$

c) $\langle u| = |u\rangle^{\dagger} = \left[|0\rangle - 2i|1\rangle \right]^{\dagger}$

$= |0\rangle^{\dagger} + (-2i)^{*} \, |1\rangle^{\dagger}$

$= \langle 0| + 2i\langle 1|$

$\langle u|v\rangle = \left[\langle 0| + 2i\langle 1| \right]\left[i|0\rangle + |1\rangle \right]$

$= i\cancelto{1}{\langle 0|0\rangle} + 2i\,i\cancelto{0}{\langle 1|0\rangle} + \cancelto{0}{\langle 0|1\rangle} + 2i\cancelto{1}{\langle 1|1\rangle}$

$= 3i$

Matrices

A matrix acts on a vector to produce another vector. So for a vector

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

maps as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} av_0 + bv_1 \\ cv_0 + dv_1 \end{pmatrix}$$ 2 dimensional vector.

In quantum theory we will refer to such entities as operators and denote these by $\hat{M}$. Then for any ket $|v\rangle$, $\hat{M}|v\rangle$ is another ket.

Also for any bra $\langle u|$, $\langle u|\hat{M}$ is another bra.

The identity matrix is

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and one can see

$$\boxed{\text{For any } |v\rangle \qquad \hat{I}|v\rangle = |v\rangle}$$

2 Hadamard operator

Consider the operator

$$\hat{H} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

a) Let

$$|v\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + i\,|1\rangle \right)$$

and $|w\rangle = \hat{H}\,|v\rangle$. Determine an expression for $|w\rangle$ in terms of $|0\rangle$ and $|1\rangle$.

b) Let

$$\langle u| = \frac{1}{\sqrt{2}} \left(\langle 0| + i\,\langle 1| \right)$$

and $\langle w| = \langle v|\,\hat{H}$. Determine an expression for $\langle w|$ in terms of $\langle 0|$ and $\langle 1|$.

Answer: a) $|v\rangle \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

So

$$\hat{H}|v\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} = \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle$$

b) $\langle u| \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix}$

$$\langle u|\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 1-i \end{pmatrix}$$

$$\langle u|\hat{H} = \frac{1+i}{2}\langle 0| + \frac{1-i}{2}\langle 1|$$

Matrix adjoint

The transpose of a matrix flips rows and columns. So if

$$\hat{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

take this row + convert to this column

the transpose is

$$\hat{M}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

The adjoint or complex conjugate transpose is

$$\hat{M}^\dagger = \overline{M^{*T}}$$

$$= \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

This will play an important role in quantum theory. Two special classes of operators are

1) an operator $\hat{M}$ is Hermitian if

$$\hat{M}^\dagger = \hat{M}$$

2) an operator is unitary if

$$\hat{M}^\dagger \hat{M} = \hat{I}$$

3 Hermitian and unitary operators

Consider the operator

$$\hat{H} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Check whether $\hat{H}$ is Hermitian and whether it is unitary.

Answer: $\hat{H}^{+} = \hat{H}^{*T}$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}^{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \hat{H}$$

$\Rightarrow \hat{H}^{+} = \hat{H}$

so $\hat{H}$ is Hermitian.

Then $\hat{H}^{+}\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}$

$$= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \hat{H}^{+}\hat{H} = \hat{I}$