Phys 396 2026

Lecture 3

Mon: HW 1 by 5pm

Read 2.1.1.

Linear algebra

Quantum theory uses vectors, matrices and the machinery of linear algebra to describe states of quantum systems. Certain results and notions from linear algebra are important.

Complex vector spaces

The vectors required for quantum theory have complex number entries. The simplest non-trivial vectors can be represented as a column with two complex entries

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

and we denote this by a "ket"

$$\underbrace{|v\rangle}_{\text{label}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \text{complex number}$$

We can add two such kets

$$|u\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad |v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

to form a new vector

$$|\,\text{label}\,\rangle = |u\rangle + |v\rangle = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

We can multiply a ket by a scalar

$$|\,\text{new label}\rangle = \alpha |v\rangle = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$$

These operations satisfy:

1) $|u\rangle + |v\rangle = |v\rangle + |u\rangle$

2) $(|u\rangle + |v\rangle) + |w\rangle = |u\rangle + (|v\rangle + |w\rangle)$

3) There is a zero ket $|0\rangle$ s.t. for any $|u\rangle$

$$|u\rangle + |0\rangle = |u\rangle$$

4) For any numbers α, β.

$$\alpha(\beta|u\rangle) = \alpha\beta|u\rangle$$

5) $(\alpha+\beta)|u\rangle = \alpha|u\rangle + \beta|u\rangle$

6) $\alpha(|u\rangle + |v\rangle) = \alpha|u\rangle + \alpha|v\rangle$

Basis vectors (kets)

Any ket can be expressed in terms of basis kets (or vectors). The basis vectors have to satisfy certain technical requirements:

* the basis vectors are linearly independent
* the basis vectors span the space of all vectors

One set that does these is called a basis. For example the set $\{|e_1\rangle, |e_2\rangle\}$ where

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

forms a basis for the set of column vectors with two entries. Then

$$|u\rangle = u_1|e_1\rangle + u_2|e_2\rangle$$

$$u_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

The number of basis vectors is called the dimension of the vector space

So for <u>two dimensional vector spaces</u>

> There exists a basis consisting of two kets $|e_1\rangle, |e_2\rangle$ such that for any ket $|v\rangle$
>
> $$|v\rangle = v_1|e_1\rangle + v_2|e_2\rangle$$
>
> and v_1, v_2 are called the components of $|v\rangle$ in the basis $\{|e_1\rangle, |e_2\rangle\}$.
>
> These components are, for a given basis unique.

For finite dimensional vector spaces there are many possible bases and a given vector will be represented by different components in different bases.

1 Vector basis changes

Consider vectors in two dimensions and the basis

$$|e_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |e_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

Suppose that the components of a ket $|u\rangle$ in this basis are $u_1 = 3$ and $u_2 = 4i$.

a) Consider an alternative basis

$$|f_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |f_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Determine the components of $|u\rangle$ in this basis.

b) Consider an alternative basis

$$|f_1\rangle = \begin{pmatrix}2\\0\end{pmatrix} \quad \text{and} \quad |f_2\rangle = \begin{pmatrix}1\\1\end{pmatrix}.$$

Determine the components of $|u\rangle$ in this basis.

Answer: a)

$$|f_1\rangle = \frac{1}{\sqrt{2}}|e_1\rangle + \frac{1}{\sqrt{2}}|e_2\rangle$$

$$|f_2\rangle = \frac{1}{\sqrt{2}}|e_1\rangle - \frac{1}{\sqrt{2}}|e_2\rangle$$

$$\Rightarrow \quad |f_1\rangle + |f_2\rangle = \frac{2}{\sqrt{2}}|e_1\rangle \quad \Rightarrow \quad |e_1\rangle = \frac{1}{\sqrt{2}}|f_1\rangle + \frac{1}{\sqrt{2}}|f_2\rangle$$

$$|f_1\rangle - |f_2\rangle = \frac{2}{\sqrt{2}}|e_2\rangle \qquad |e_2\rangle = \frac{1}{\sqrt{2}}|f_1\rangle - \frac{1}{\sqrt{2}}|f_2\rangle$$

Thus $\quad |u\rangle = 3|e_1\rangle + 4i|e_2\rangle$

$$= \frac{3}{\sqrt{2}}|f_1\rangle + \frac{3}{\sqrt{2}}|f_2\rangle + \frac{4i}{\sqrt{2}}|f_1\rangle - \frac{4i}{\sqrt{2}}|f_2\rangle$$

$$= \frac{3+4i}{\sqrt{2}}|f_1\rangle + \frac{3-4i}{\sqrt{2}}|f_2\rangle$$

and in the basis $\{|f_1\rangle, |f_2\rangle\}$ the components are:

$$\frac{3+4i}{\sqrt{2}} \qquad \frac{3-4i}{\sqrt{2}}$$

b) $|f_1\rangle = 2|e_1\rangle \Rightarrow |e_1\rangle = \frac{1}{2}|f_1\rangle$

$|f_2\rangle = |e_1\rangle + |e_2\rangle \Rightarrow |f_2\rangle = \frac{1}{2}|f_1\rangle + |e_2\rangle$

$\Rightarrow |e_2\rangle = |f_2\rangle - \frac{1}{2}|f_1\rangle$

Thus

$$|e_1\rangle = \frac{1}{2}|f_1\rangle$$

$$|e_2\rangle = |f_2\rangle - \frac{1}{2}|f_1\rangle$$

Then

$$|u\rangle = 3|e_1\rangle + 4i|e_2\rangle$$

$$= \frac{3}{2}|f_1\rangle + 4i|f_2\rangle - \frac{4i}{2}|f_1\rangle$$

$$= \frac{3-4i}{2}|f_1\rangle + 4i|f_2\rangle$$

and the components are

$$\frac{3-4i}{2}, \quad 4i$$

Bra vectors

It is also useful to consider row vectors in the mathematics of quantum Theory. These have form $(v_1 \;\; v_2)$ where v_1 and v_2 are complex numbers. In this treatment

1) row vectors are distinct from column vectors.
2) one cannot add a raw vector to a column vector.

The column vectors are called "bra" vectors and are denoted

$$\langle \underset{\uparrow}{v} | = (\underset{\uparrow}{v_1} \;\; \underset{\uparrow}{v_2})$$

label

complex numbers

The rules for column vectors apply to row vectors. Thus we can add

$$\langle u| + \langle v| = (u_1 \;\; u_2) + (v_1 \;\; v_2) = (u_1+v_1 \;\; u_2+v_2)$$

and multiply by a scalar

$$\alpha\langle u| = \alpha(u_1 \;\; u_2) = (\alpha u_1 \;\; \alpha u_2)$$

These satisfy the same abstract properties as column vectors or kets.

Action of a bra on a ket

A bra vector can act on a ket to produce a number. So

$$\langle u| = (u_1 \;\; u_2)$$

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\langle u||v\rangle = (u_1 \;\; u_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2$$

"$\langle u|$ acts on $|v\rangle$"

and we write the action as

$\langle u|v\rangle$.

This action is linear. So

$$1) \quad \langle u|\big(|v\rangle + |w\rangle\big) = \langle u|v\rangle + \langle u|w\rangle$$

$$2) \quad \langle u|\big(\alpha|v\rangle\big) = \alpha\langle u|v\rangle$$

Similarly

$$1) \quad \big(\langle u| + \langle v|\big)|w\rangle = \langle u|w\rangle + \langle v|w\rangle$$

$$2) \quad \big(\alpha\langle u|\big)|w\rangle = \alpha\langle u|w\rangle$$

Mapping kets to bras + inner product

Suppose we have a ket $|u\rangle = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. We can map this to a bra as follows

$$\begin{array}{ccc} |u\rangle & \longrightarrow & \langle u| \\ \| & & \| \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & & \begin{pmatrix} u_1^* & u_2^* \end{pmatrix} \end{array}$$

This involves complex conjugation and the transpose operation of matrices, and is called the "complex conjugate transpose"

$$\begin{pmatrix} u_1^* & u_2^* \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^\dagger$$

or

$$\langle u| = |u\rangle^\dagger$$

Then we define

> Given two kets $|u\rangle$, $|v\rangle$ the inner product of $|u\rangle$ with $|v\rangle$ is
>
> $$\langle u|v\rangle$$

This satisfies:

$$\langle u|v\rangle = (\langle v|u\rangle)^*$$

plus linearity in $|v\rangle$.

<u>Orthogonality and normalization</u>

The magnitude of a ket $|v\rangle$ can be determined by $\langle v|v\rangle$ which we can show is positive. Then

The <u>magnitude</u> of $|v\rangle$ is $\sqrt{\langle v|v\rangle}$

We define

> A ket is normalized if $\langle v|v\rangle = 1$

and

> Two kets $|u\rangle, |v\rangle$ are orthogonal if $\langle u|v\rangle = 0$

<u>Orthonormal basis</u>

A set of kets is called an orthonormal basis $\Leftrightarrow$ the kets are all normalized and all orthogonal to each other.

2 Inner product

Let

$$|u\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad |v\rangle = \begin{pmatrix} 3+4i \\ 3-4i \end{pmatrix}.$$

a) Determine the inner product $\langle u|v\rangle$.

b) Determine the inner product $\langle v|u\rangle$.

c) Check that $\langle v|u\rangle = (\langle u|v\rangle)^*$.

Answer:

a) $\langle u| = (1^* \; (-i)^*) = (1 \; i)$

$$\langle u|v\rangle = (1 \; i)\begin{pmatrix} 3+4i \\ 3-4i \end{pmatrix} = 3+4i + i(3-4i)$$

$$= 3+4i+3i \underbrace{-4i^2}_{+4}$$

$$= 7+7i$$

b) $\langle v| = (3-4i \quad 3+4i)$

$$\langle v|u\rangle = (3-4i \quad 3+4i)\begin{pmatrix} 1 \\ -i \end{pmatrix} = 3-4i - i(3+4i)$$

$$= 3-4i-3i+4$$

$$= 7-7i$$

c) Yes $(\langle u|v\rangle)^* = 7-7i = \langle v|u\rangle$

3 Vector bases

Consider the following vectors in two dimensions

$$|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad |g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

a) Determine the inner product of each vector with itself, i.e. $\langle f|f\rangle$.

b) Determine the inner product $\langle f|g\rangle$.

c) Do these constitute an orthonormal basis?

Answer: a)

$$\langle f| = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \end{pmatrix}$$

$$\langle f|f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{1}{2}(-i^2+1) = \frac{1}{2}2 = 1$$

$$\langle g| = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \end{pmatrix}$$

$$\langle g|g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2}(-i^2+1) = 1$$

b)

$$\langle f|g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{2}((-i)^2+1) = 0$$

c) Yes.