Quantum Information: Homework 1

Due: 25 August 2026

1 Binary addition

For each of the following pairs of numbers, represent each number in binary form, add the numbers using binary form and convert the result to a decimal number. Verify that the procedure was correct.

a) 9 and 11.
b) 35 and 124.
c) 76 and 85.

2 Modular addition

The following problems involve identities with modular addition. To prove them, consider a single binary digit, a, and prove these by checking that they are true for all possible choices of a.

a) Show that $a \oplus a = 0$.
b) Show that $a \oplus (1 \oplus a) = 1$.

3 Binary addition

Consider two numbers in binary form

$$a = a_{n-1}a_{n-2}\dots a_1a_0$$
$$b = b_{n-1}b_{n-2}\dots b_1b_0$$

Let $s = a + b$. This will have form

$$s = s_n s_{n-2}\dots s_1 s_0$$

and the question is how to determine the binary digits of s from those of a and b. Clearly the digit s_k will depend on a_k and b_k. But it will also depend on the "carry," c_k that from the addition on the $k-1$ bit.

a) Show, by checking all possible bit values, that

$$s_k = a_k \oplus b_k \oplus c_k. \tag{1}$$

b) Show, by checking all possible bit values, the carry from the addition of the k bits is

$$c_{k+1} = a_kb_k \oplus c_ka_k \oplus c_kb_k. \tag{2}$$

c) Use Eqs (1) and (2) to add $1011 + 1101$. Note that the carry on the bit zero is automatically $c_0 = 0$. Verify that the result is correct.

d) Suppose that two n bit numbers need to be added. Counting each modular addition of two bits as one operation and each multiplication of two single bits as one operation, determine the total number of such operations needed to add two n bit numbers.

4 Gates and modular arithmetic

Modular addition of bits and multiplication of bits can be used to describe the output of basic gates.

a) Show, by checking all possible inputs, that two successive not gates map

$$a \mapsto a.$$

b) Show, by checking all possible inputs, that the OR gate maps

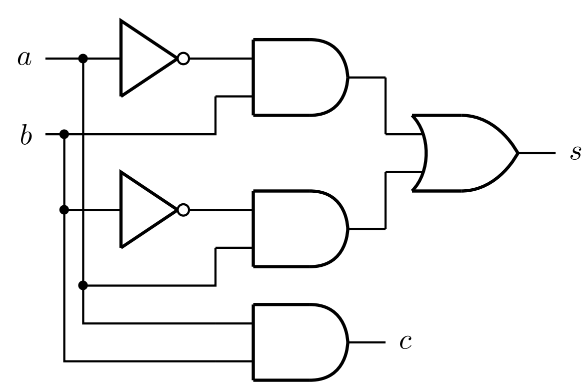
$$(a, b) \mapsto 1 \oplus (1 \oplus a)(1 \oplus b).$$

If you wonder how to obtain a rule like this, it comes from DeMoivre's laws.

c) Provide the circuit that constructs an XOR gate from AND, OR and NOT gates. Show that the circuit is correct.

5 Half-adder

Suppose that a and b are two binary digits. Consider the illustrated half-adder circuit.



a) Show that this produces the sum $s = a \oplus b$.

b) Show that this produces the carry $c = ab$.

6 Multiplying binary numbers

Consider two bit binary numbers

$$x = x_1 x_0 \quad \text{and}$$
$$y = y_1 y_0.$$

Then the product $z = xy$ has from $z = z_3 z_2 z_1 z_0$ where

$$z_3 = x_1 x_0 y_1 y_0$$
$$z_2 = (x_1 y_1)(1 \oplus x_0 y_0)$$
$$z_1 = x_1 y_0 \oplus x_0 y_1$$
$$z_0 = x_0 y_0$$

a) Check for $x = 0$ and $y = 0, 1, 2, 3$ that the formulas for z_k are correct.

b) Check for $x = 1$ and $y = 1, 2, 3$ that the formulas for z_k are correct.

c) Check for $x = 2$ and $y = 2, 3$ that the formulas for z_k are correct.

d) Check for $x = 3$ and $y = 3$ that the formulas for z_k are correct.

e) Provide circuits using AND and XOR gates for determining each bit of z.