Thurs: Seminar

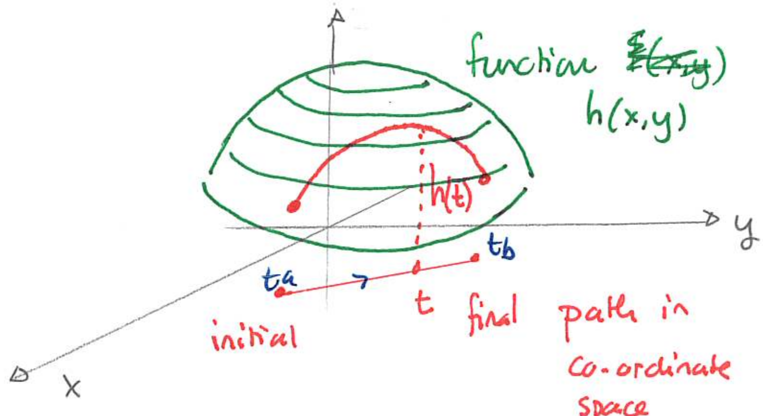
Fri: HW3 5pm Corrections HW 1

Tues: Corrections HW 2 HW 4 due

Read 1.3.1 surface int. 1.3.4. 1.3.5

Line integrals

A line integral aggregates information about a function in three dimensions along some line through space. For example we could imagine an altitude profile $h(x,y)$ above a two dimensional plane.



We could traverse a path in the plane and ask what the average altitude along the path is. To do this, conceptually

1. Describe the altitude function $h(x,y)$
2. Describe the trajectory in x,y space. We parameterize the path by

$$t_a \leq t \leq t_b \quad \rightsquigarrow \quad x(t), y(t).$$

3. Along the trajectory the height is a function of t

$$h(t) = h(x(t), y(t))$$

4. The average height is

$$\bar{h} = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} h(x(t), y(t))\, dt$$

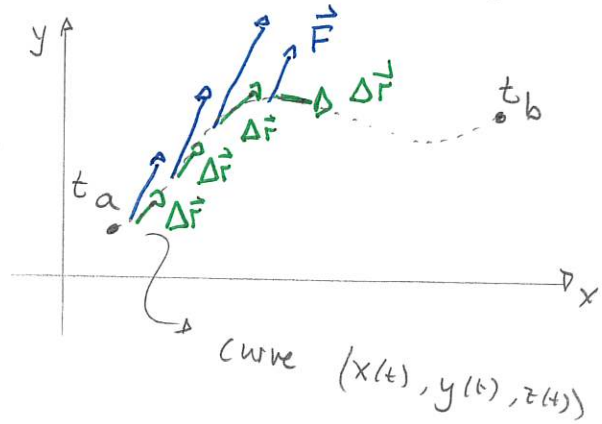
This is a <u>line integral</u> of a scalar function.

Electromagnetic theory will require line integrals of vector functions. We could define these in many ways but we want a result that is

1) independent of the Cartesian co-ordinate system used.
2) independent of the parameterization of the curve.

One particularly useful example is related to the mechanical notion of work.

Consider a particle that follows a trajectory parameterized by $t_a \leq t \leq t_b$. Let $\vec{F} = \vec{F}(x,y,z)$ be the force exerted on the particle. Then the work done by $\vec{F}$ is



$$W = \sum_{\text{all segments}} \vec{F}\cdot\Delta\vec{r}$$

Aggregates $\vec{F}\cdot\Delta\vec{r}$ along curve

Now

$$\Delta\vec{r} = \Delta x\,\hat{\imath} + \Delta y\,\hat{\jmath} + \Delta z\,\hat{k}$$

$$\cong \left[\frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k}\right]\Delta t$$

and

$$W = \sum \vec{F}\cdot\left[\frac{\Delta x}{\Delta t}\hat{\imath} + \frac{\Delta y}{\Delta t}\hat{\jmath} + \frac{\Delta z}{\Delta t}\hat{k}\right]\Delta t$$

As $\Delta t \to 0$

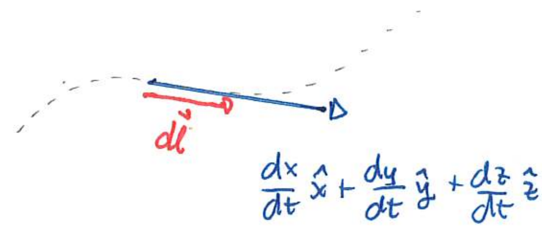
$$W = \int_{t_a}^{t_b} \vec{F}\cdot\left[\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right]dt$$

co-ordinate label.

$$= \int_{t_a}^{t_b} \underbrace{\left[F_x(x(t),y(t),z(t))\frac{dx}{dt} + F_y(x(t),y(t),z(t))\frac{dy}{dt} + F_z(x(t),y(t),z(t))\frac{dz}{dt}\right]}_{\text{Function of } t \text{ constructed from } \vec{F} \text{ and trajectory.}} dt$$

Now

$$\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$



is a vector tangent to the trajectory. So

$$\vec{dl} = \left[\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right] dt \equiv dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

is also tangent to the trajectory. We could then express the work as

$$W = \int_{\text{trajectory}} \vec{F}\cdot\vec{dl}$$

This is an example of a line integral.

Consider the vector field

$$\vec{V} = V_x(x,y,z)\,\hat{x} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$

and the trajectory described by.

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z} \qquad \text{for} \qquad t_a \le t \le t_b$$

Then the line integral of $\vec{V}$ along $\vec{r}(t)$ is

$$\int \vec{V}\cdot\vec{dl} := \int_{t_a}^{t_b} \left[V_x(x(t),y(t),z(t))\frac{dx}{dt} + V_y(x(t),y(t),z(t))\frac{dy}{dt} + V_z(\dots)\frac{dz}{dt}\right] dt$$

where the derivatives are evaluated at $x(t), y(t), z(t)$

One can show

1) the line integral is independent of the parameterization of the curve

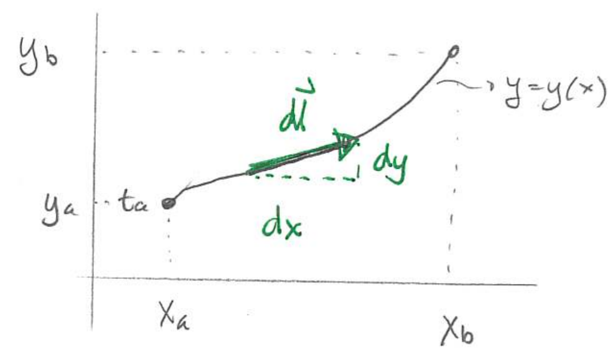
2) the line integral is independent of the co-ordinate system used.

However,

$\boxed{\text{In general the line integral between two points will depend on the path or trajectory between the points.}}$

Parameterization of the curve

In general any parameterization of the curve gives the same output. However, the can be convenient parameterizations in terms of co-ordinates. Suppose that each value of x corresponds to one value of y. Then the curve can be parameterized by x



$$d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= dx\,\hat{x} + \frac{dy}{dx}dx\,\hat{y} + \frac{dz}{dx}dx\,\hat{z}$$

and

$$\vec{v}\cdot d\vec{l} = v_x\,dx + v_y\frac{dy}{dx}dx + v_z\frac{dz}{dx}dx = \left[v_x + v_y\frac{dy}{dx} + v_z\frac{dz}{dx}\right]dx$$

$$\Rightarrow \int \vec{v}\cdot d\vec{l} = \int_{x_a}^{x_b}\left[v_x(x,y(x),z(x)) + v_y\frac{dy}{dx} + v_z\frac{dz}{dx}\right]dx$$

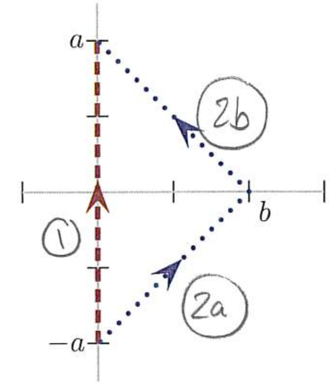
and this can simplify calculations

1 Line integral in two dimensions: straight segments, 1

Let $\mathbf{v} = -y\hat{\mathbf{x}}$. Determine

$$\int \mathbf{v} \cdot d\mathbf{l}$$

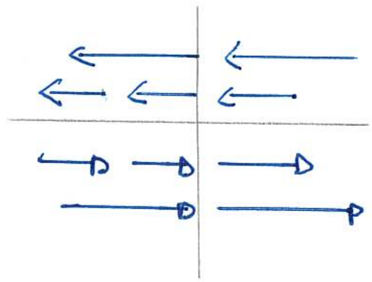
along each of the two illustrated paths.



The vector field has components

$V_x = -y$

$V_y = 0$



Along path 1

$$\begin{matrix} x = 0 \\ -a \leq y \leq a \end{matrix} \Rightarrow \text{ use } y \text{ as a parameter}$$

$$\vec{dl} = dx\hat{x} + dy\hat{y} = \cancel{\frac{dx}{dy}\,dy\,\hat{x}} + dy\hat{y} = dy\,\hat{y}$$

Then $\vec{v} \cdot \vec{dl} = -y\hat{x} \cdot dy\hat{y} = 0 \quad \Rightarrow \quad \int_1 \vec{v} \cdot \vec{dl} = 0$

This is sensible because the component of the vector along the path is zero

Along path 2. Split into two parts

$$\int_2 \vec{v} \cdot \vec{dl} = \int_{2a} \vec{v} \cdot \vec{dl} + \int_{2b} \vec{v}\, \vec{dl}$$

lower diagonal $\quad 0 \leq x \leq b$

$$y = \frac{a}{b}x - a$$

We can use x as a parameter. Then

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} = dx\,\hat{x} + \overbrace{\frac{dy}{dx}}^{a/b}\,\hat{y}\,dx$$

$$= dx\,\hat{x} + \frac{a}{b}\,dx\,\hat{y}$$

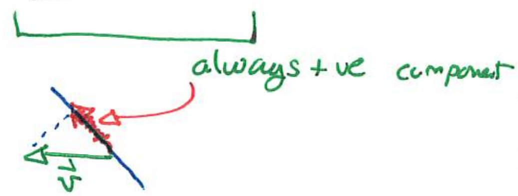
$$\text{So} \quad \vec{v}\cdot\vec{dl} = -y\hat{x}\cdot\left[\hat{x} + \frac{a}{b}\hat{y}\right]dx$$

$$= -y\,dx$$

$$= -\left(\frac{a}{b}x - a\right)dx$$

$$\int_{2a}\vec{v}\cdot\vec{dl} = -\int_0^b\left(\frac{a}{b}x - a\right)dx = -\left[\frac{a}{b}\frac{x^2}{2} - ax\right]_0^b$$

$$= -\left[\frac{ab}{2} - ab\right] = \frac{ab}{2} \quad \Rightarrow \quad \int_{2a}\vec{v}\cdot\vec{dl} = \frac{ab}{2}$$



Upper diagonal   Invert the path:

$$\int_{\nwarrow}\vec{v}\cdot\vec{dl} = -\int_{\searrow}\vec{v}\cdot\vec{dl}$$

For the inverted path $\quad 0 \leq x \leq b$

$$y = -\frac{a}{b}x + a$$

$$\Rightarrow \vec{dl} = dx\,\hat{x} + dy\,\hat{y} = dx\,\hat{x} + \frac{dy}{dx}dx\,\hat{y} = dx\,\hat{x} - \frac{a}{b}\,dx\,\hat{y}$$

$$\text{So} \quad \vec{v}\cdot\vec{dl} = -y\,dx = -\left[-\frac{a}{b}x + a\right]dx$$

$$= \left(\frac{a}{b}x - a\right)dx$$

$$\int_{\searrow}\vec{v}\cdot\vec{dl} = \int_0^b\left(\frac{a}{b}x - a\right)dx = \left[\frac{ax^2}{2b} - ax\right]\Big|_0^b = -\frac{ab}{2}$$

Thus

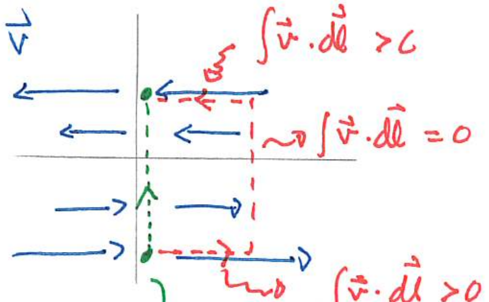
$$\int_{2b}\vec{v}\cdot\vec{dl} = \frac{ab}{2}$$

Thus $\quad \displaystyle\int_2 \vec{v}\cdot\vec{dl} = ab$

The example illustrates

Line integrals depend on the initial and final points of the path and can depend on the path.

In the case of $\vec{v} = -y\hat{x}$



$\vec{v}$ perpendicular to path $\int \vec{v} \cdot d\vec{l} = 0$

Integration theorems for line integrals

Calculus relates integration to differentiation via the fundamental theorem of calculus

> For any suitably well-behaved function $f(x)$
>
> $$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

This allows one to evaluate

$$\int_a^b g(x)\,dx$$

if one can find $f(x)$ so that $g(x) = \frac{df}{dx}$. Here f is the antiderivative of g

Does such a result exist for line integration. We will see:

1) it is not always possible to evaluate a line integral via an antiderivative.

2) there is a simple condition that describes when it is possible to evaluate a line integral via an antiderivative

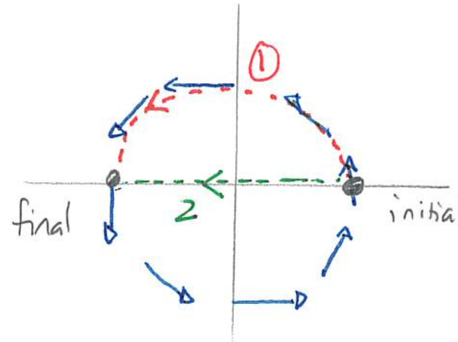
So we ask

> Given $\vec{v}$ and a trajectory $x(t), y(t), z(t)$ is there a function $g(x,y,z)$ such that
>
> $$\int_{t_i}^{t_f} \vec{v} \cdot d\vec{l} = g(x_f, y_f, z_f) - g(x_i, y_i, z_i)$$
>
> where $x_f = x(t_f)$ .... $x_i = x(t_i)$
>
> $y_f = y(t_f)$ $y_i = y(t_i)$

If this is true then it requires that the line integral only depends on the initial + final location on the path. This is not always the case. For example,

with

$$\vec{v} = -y\hat{x} + x\hat{y}$$



and the illustrated initial + final locations

$$\int_{\text{path } 1} \vec{v} \cdot d\vec{l} > 0 \quad \text{and} \quad \int_{\text{path } 2} \vec{v} \cdot d\vec{l} = 0$$

So there cannot be a function g s.t. $\int \vec{v} \cdot d\vec{l} = g_f - g_i$ as this would give the same result for both paths

<u>Fundamental theorem for gradients</u>

Suppose that there is a function $g(x,y,z)$ such that

$$\vec{v} = \vec{\nabla} g$$

Then

$$\vec{v} = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

and

$$\int_{t_i}^{t_f} \vec{v} \cdot d\vec{l} = \int_{t_i}^{t_f} \left[\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} \right] dt$$

$$= \int_{t_i}^{t_f} \frac{d}{dt} g(x(t), y(t), z(t))\, dt$$

$$= g(x(t_f), y(t_f), z(t_f)) - g(x(t_i), y(t_i), z(t_i))$$

$$= g(\text{end path}) - g(\text{start of path})$$

This gives the <u>fundamental theorem for gradients</u>:

> Given a vector field $\vec{v}$, there exists a function g such that $\vec{v} = \vec{\nabla} g$ if and only if
>
> $$\int_{\text{path}} \vec{v} \cdot d\vec{l} = \int_{\text{path}} \vec{\nabla} g \cdot d\vec{l} = g(b) - g(a)$$
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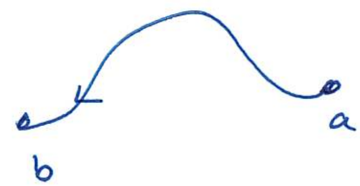
> 
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>
> where $g(b) = g(x,y,z)\Big|_{\text{evaluate at } b}$
>
> $g(a) = g(x,y,z)\Big|_{\text{evaluate at } a}$

We proved that if $\vec{v} = \vec{\nabla} g$ then $\int \vec{v} \cdot d\vec{l} = \ldots$ Consider the reverse. Suppose there is a function $g(x,y,z)$ s.t.

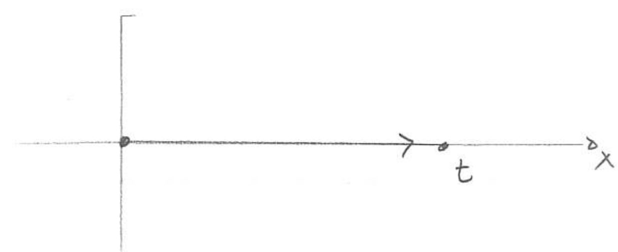
$$\int \vec{v} \cdot d\vec{l} = g(b) - g(a)$$

for any path. Consider the path

$x(s) = s \qquad 0 \le s \le t$

$y(s) = 0$

$z(s) = 0$



Then

$$\int \vec{v} \cdot d\vec{l} = g(x(t), 0, 0) - g(0,0,0)$$

$$\int_0^t \left[V_x \overset{1}{\frac{dx}{ds}} + \cancel{V_y \frac{dy}{ds} + V_z \frac{dz}{ds}} \right] ds = g(x(t), 0, 0) - g(0,0,0)$$

$$\Rightarrow \int_0^t V_x \, ds = g(x(t), 0, 0) \ldots$$

So $$\frac{d}{dt} \int_0^t V_x \, ds = \frac{\partial g}{\partial x} \frac{dx}{dt} \Rightarrow V_x(x(t), y(t), z(t)) = \frac{\partial g}{\partial x}$$

etc... $\Rightarrow \vec{v} = \vec{\nabla} g$

Consequences of the fundamental theorem for gradients are:

$\boxed{\text{If } \vec{v} = \vec{\nabla} g \text{ then } \int \vec{v}\cdot d\vec{l} \text{ only depends on the initial and final points of the path}}$

If $\vec{v} = \vec{\nabla} g$ then around any closed loop

$$\oint \vec{v}\cdot d\vec{l} = 0$$

If $\oint \vec{v}\cdot d\vec{l} = 0$ then $\vec{v} = \vec{\nabla} g$.

Separately if $\vec{v} = \vec{\nabla} g = 0$ then $\vec{\nabla}\times\vec{v} = 0$. This gives.

There exists $g(x,y,z)$ such that

$$\int_a^b \vec{v}\cdot d\vec{l} = g(b) - g(a)$$

for any path $\Longleftrightarrow$ $\vec{\nabla}\times\vec{v} = 0$

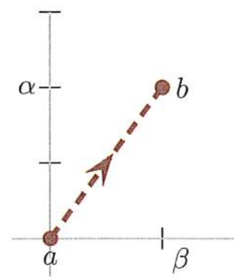
2 Fundamental theorem for gradients, 1

Let $g(x, y, z) = \tau x^2 y$ where τ is constant.

a) Determine $\mathbf{v} := \boldsymbol{\nabla} g$.

b) Determine $\int \mathbf{v} \cdot \mathrm{d}\mathbf{l}$ along the illustrated path.

c) Determine $g(b) - g(a)$ and check that the fundamental theorem for gradients is valid.



Answer: a) $\vec{v} = \vec{\nabla} g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z} = 2\tau x y \hat{x} + \tau x^2 \hat{y}$

b) Here $0 \leq x \leq \beta$

$y = \frac{\alpha}{\beta} x$

$$\vec{dl} = dx\hat{x} + dy\hat{y}$$
$$= dx\hat{x} + \frac{dy}{dx} dx \hat{y}$$
$$= dx\hat{x} + \frac{\alpha}{\beta} dx \hat{y}$$

So $\vec{v} \cdot \vec{dl} = 2\tau x y\, dx + \frac{\alpha}{\beta}\tau x^2 dx$

$= \tau\left(2xy + \frac{\alpha}{\beta}x^2\right)dx$

But $y = \alpha/\beta\, x$

$\Rightarrow \vec{v} \cdot \vec{dl} = \tau\left(2\frac{\alpha}{\beta}x^2 + \frac{\alpha}{\beta}x^2\right) dx = 3\tau\frac{\alpha}{\beta}x^2 dx$

So $\int_a^b \vec{v} \cdot \vec{dl} = 3\tau\frac{\alpha}{\beta}\int_0^\beta x^2 dx = \tau\frac{\alpha}{\beta}x^3 \Big|_0^\beta \Rightarrow \int \vec{v} \cdot \vec{dl} = \tau\alpha\beta^2$

c) $g(b) = \tau\beta^2\alpha$

$g(a) = 0$

$\Rightarrow g(b) - g(a) = \tau\alpha\beta^2 = \int \vec{v}\, \vec{dl}$

WORKS!