Tues: HW 2 5pm

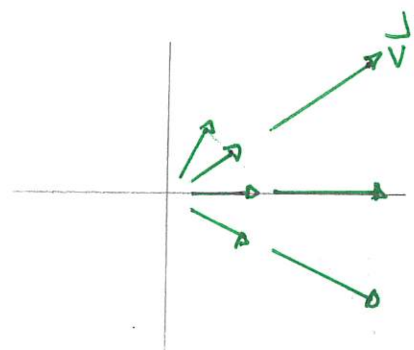
Fri: HW 3 5pm

Thurs: Read. 1.3.1, 1.3.3

Divergence of a vector field

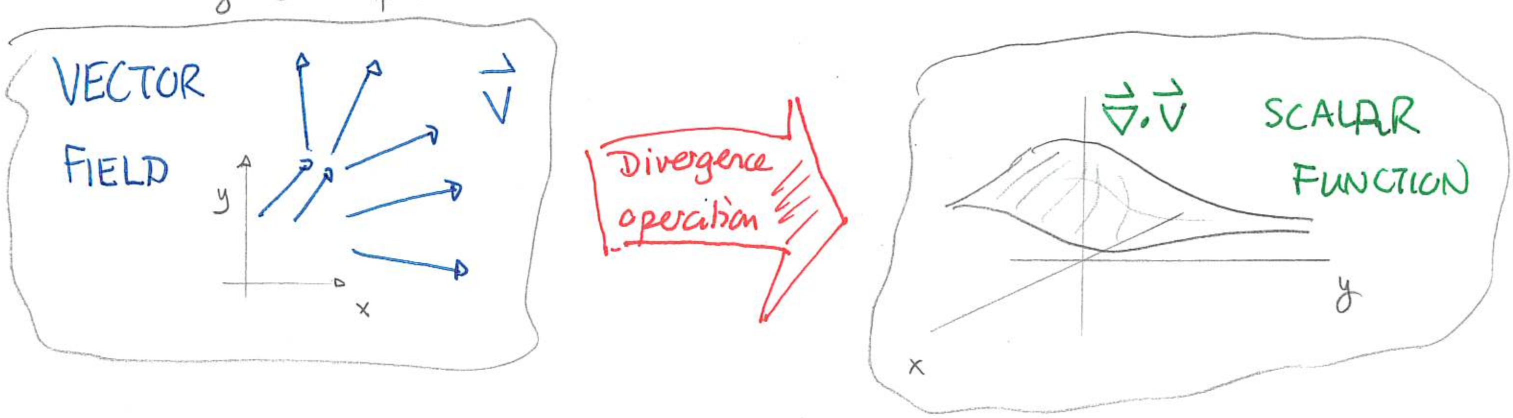
In general a vector field can vary from one location to another. We should be able to differentiate vector fields. One example, the divergence captures the degree to which a field diverges.

Consider the illustrated field. We see that



$$\frac{\partial V_x}{\partial x} > 0 \quad \text{and} \quad \frac{\partial V_y}{\partial y} > 0$$

The divergence maps.



The operation is defined as

> Let x, y, z be Cartesian co-ordinates. Then if $\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$
>
> $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Note:

1) Divergence produces a scalar function - there cannot be any basis vectors remaining

2) The divergence formula only applies to Cartesian co-ordinates. It returns the same result regardless of the particular Cartesian co-ordinate system.

3) The divergence formula must be modified for systems where the basis vectors change from one location to another.

4) Divergence is linear. If α, β are constants and $\vec{u}, \vec{v}$ any vector field

$$\vec{\nabla}\cdot(\alpha\vec{u}+\beta\vec{v}) = \alpha\vec{\nabla}\cdot\vec{u}+\beta\vec{\nabla}\cdot\vec{v}$$

The divergence of a vector field can be regarded as the action of $\vec{\nabla}$ on $\vec{V}$

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(V_x\hat{x}+V_y\hat{y}+V_z\hat{z}\right)$$

$$= \hat{x}\frac{\partial}{\partial x}\cdot(V_x\hat{x}) + \hat{x}\frac{\partial}{\partial x}\cdot(V_y\hat{y}) + \dots$$

$$= \overset{1}{\cancel{\hat{x}\cdot\hat{x}}}\frac{\partial V_x}{\partial x} + \overset{0}{\cancel{\hat{x}\cdot\hat{y}}}\frac{\partial V_y}{\partial x} + \dots$$

$$= \frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$$

Electromagnetic Theory I: Class 3

25 August 2026

1 Divergence, 2

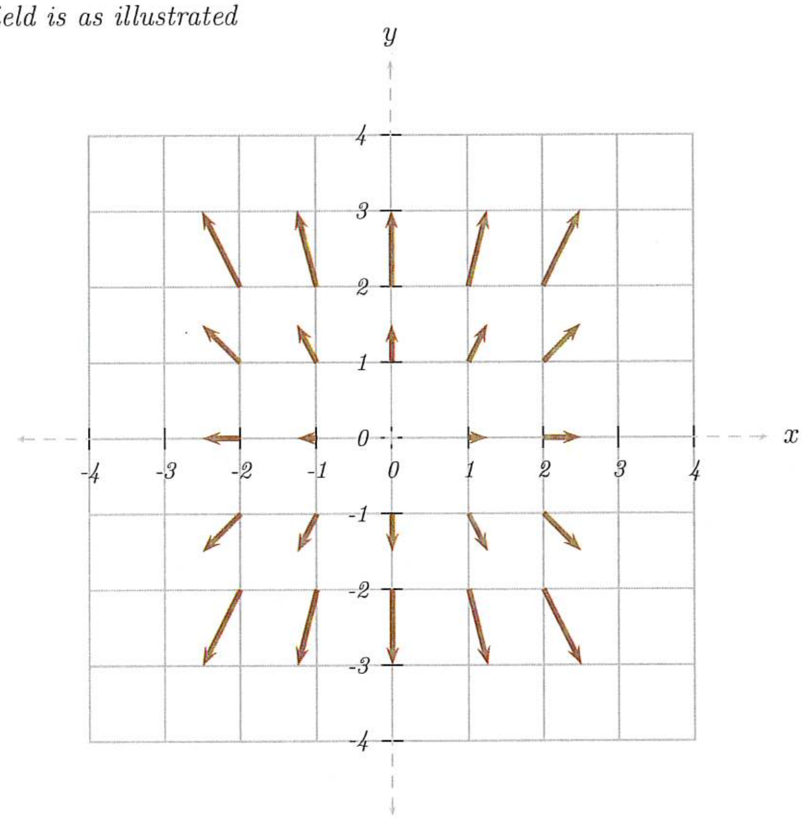
For each of the following, **sketch** the vector field and **determine its divergence.**

a) $\mathbf{v} = \dfrac{x}{4}\,\hat{\mathbf{x}} + \dfrac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \dfrac{-y}{2}\,\hat{\mathbf{x}} + \dfrac{x}{2}\,\hat{\mathbf{y}}$

Answer:

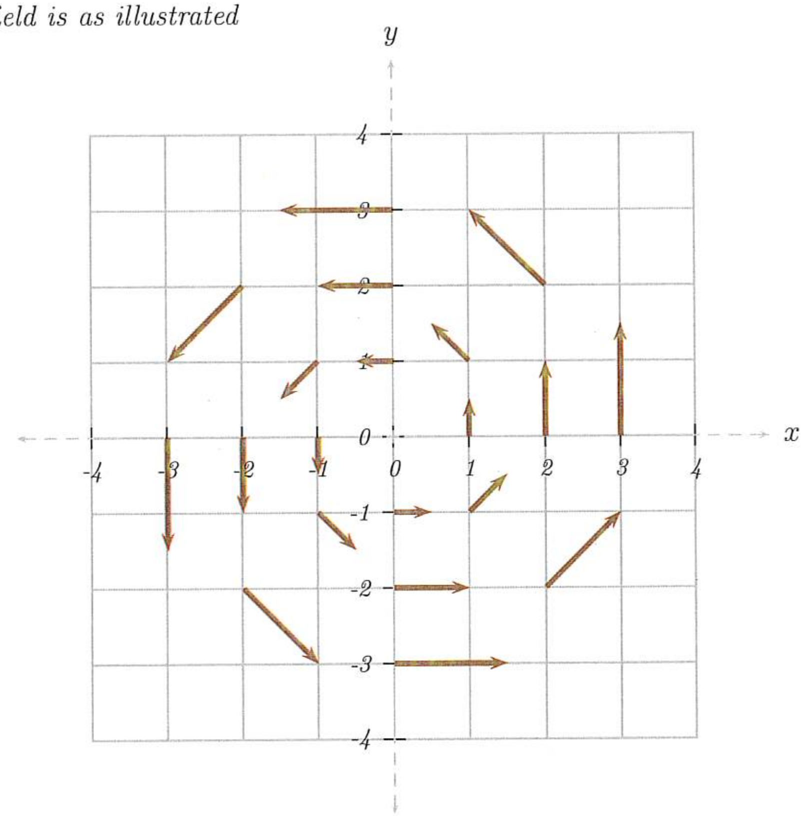
a) The vector field is as illustrated



The components are $v_x = x/4$ and $v_y = y/2$. The divergence is

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{x^2}{4}\right) + \frac{\partial}{\partial y}\left(\frac{y}{2}\right) \\
&= \frac{3}{4}.
\end{aligned}$$

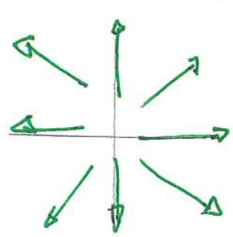
b) The vector field is as illustrated



The components are $v_x = -y/2$ and $v_y = x/2$. The divergence is

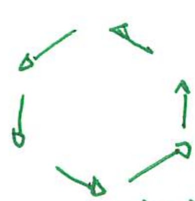
$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(-\frac{y}{2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{2}\right) \\
&= 0.
\end{aligned}$$

In general we find (although there can be exceptions)



$\vec{\nabla} \cdot \vec{v} > 0$

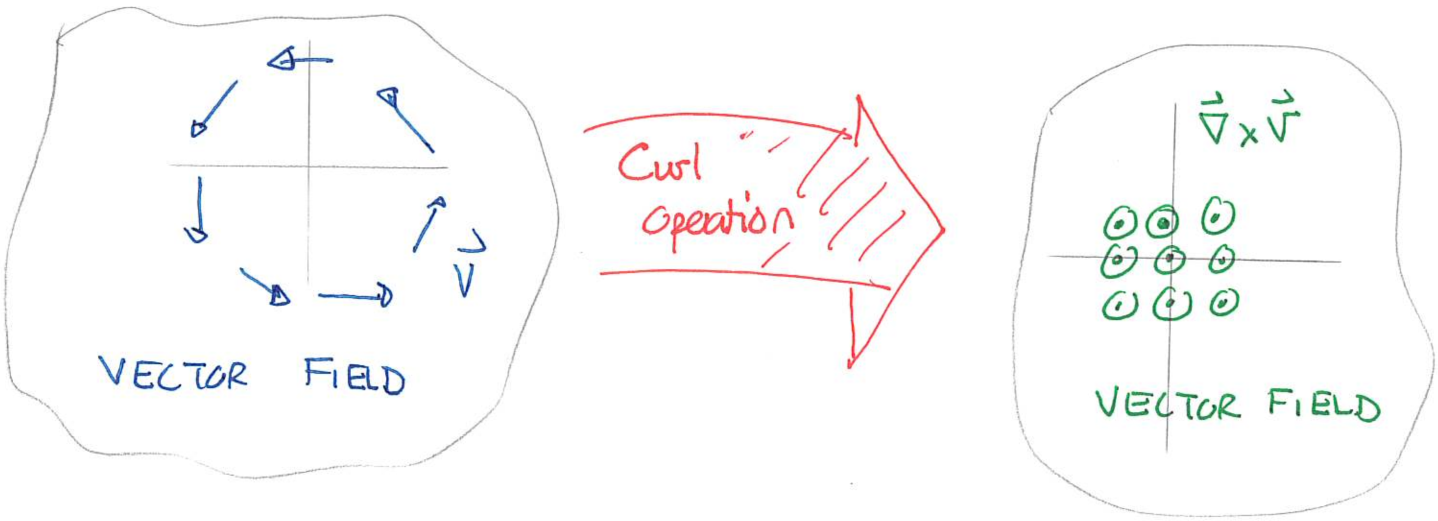
$\vec{\nabla} \cdot \vec{v} < 0$



$\vec{\nabla} \cdot \vec{v} = 0$

Curl of a vector field

Rotational aspects of a field are capture by the curl. which maps



This is defined via:

Let x, y, z be Cartesian co-ordinates and $\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$. Then the curl of $\vec{V}$ is

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{x} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{y} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z}$$

Note: 1) Curl produces vectors and there must be unit vectors present after the operation

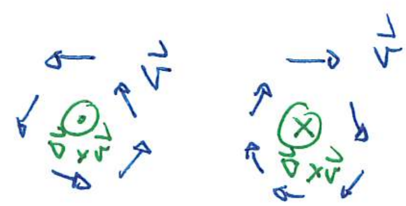
2) The curl formula only applies to Cartesian co-ordinates and it gives the same esult for any Cartesian system.

3) The curl formula must be modified for non-Cartesian co-ordinate systems.

4) For any numbers α, β and vectors $\vec{u}, \vec{v}$

$$\vec{\nabla} \times (\alpha \vec{u} + \beta \vec{v}) = \alpha \vec{\nabla} \times \hat{u} + \beta \vec{\nabla} \times \vec{v}$$

5) The direction of the curl is associated with a right-hand rule



2 Curl examples

Determine the curl of:

a) $\mathbf{v} = \frac{x}{4}\,\hat{\mathbf{x}} + \frac{y}{2}\,\hat{\mathbf{y}}$

b) $\mathbf{v} = \frac{-y}{2}\,\hat{\mathbf{x}} + \frac{x}{2}\,\hat{\mathbf{y}}$

Answer:

a) Here

$$\begin{aligned} v_x &= \frac{x}{4} \\ v_y &= \frac{y}{2} \end{aligned}$$

Thus

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{4} & \frac{y}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{x}{4}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{y}{2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{4}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\,[0-0] \\ &= 0. \end{aligned}$$

b) Here

$$\begin{aligned} v_x &= -\frac{y}{2} \\ v_y &= \frac{x}{2} \end{aligned}$$

Thus

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{2} & \frac{x}{2} & 0 \end{vmatrix} \\ &= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{2}\right)\right] + \hat{\mathbf{y}}\left[\frac{\partial}{\partial z}\left(\frac{-y}{2}\right) - \frac{\partial}{\partial x}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{2}\right)\right] \\ &= \hat{\mathbf{x}}\,[0-0] + \hat{\mathbf{y}}\,[0-0] + \hat{\mathbf{z}}\left[\frac{1}{2} + \frac{1}{2}\right] \\ &= \hat{\mathbf{z}}. \end{aligned}$$

Vector derivatives of sums + products

The various vector derivatives are linear in the following sense:

1) Gradient $\quad \vec{\nabla}(f+g) = \vec{\nabla}f + \vec{\nabla}g$

2) Divergence $\quad \vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{v}$

3) Curl $\quad \vec{\nabla}\times(\vec{u}+\vec{v}) = \vec{\nabla}\times\vec{u} + \vec{\nabla}\times\vec{v}$

Multiplication rules can be derived from the product rule for differentiation.

1) gradient

$$\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f)$$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$$

Note that $(\vec{A}\cdot\vec{\nabla})\vec{B} = A_x\frac{\partial}{\partial x}\vec{B} + A_y\frac{\partial}{\partial y}\vec{B} + A_z\frac{\partial}{\partial z}\vec{B}$

$$= A_x\frac{\partial}{\partial x}(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + A_y\frac{\partial}{\partial y}(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + \dots$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = \left(A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right)\hat{x} + \left(A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right)\hat{y} + \left(A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right)\hat{z}$$

In general $(\vec{A}\cdot\vec{\nabla})\vec{B} \neq (\vec{\nabla}\cdot\vec{A})\vec{B}$

↳ differentiates $\vec{B}$ ↳ differentiates $\vec{A}$

2) divergence

$$\vec{\nabla}\cdot(f\vec{A}) = (\vec{\nabla}f)\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

3) curl See section 1.2.6

3 Divergence of a dot product, 2

In general

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

a) Verify that this is true for

$$\mathbf{A} = x\cancel{\hat{\mathbf{z}}}\hat{x}$$
$$\mathbf{B} = y\hat{\mathbf{x}}$$

b) Check that, for these vectors, $(\mathbf{B}\cdot\nabla)\mathbf{A} \neq (\nabla\cdot\mathbf{B})\mathbf{A}$.

Answer: a) Here $\vec{A}\cdot\vec{B} = xy$

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \frac{\partial}{\partial x}(xy)\,\hat{x} + \frac{\partial}{\partial y}(xy)\,\hat{y} + \frac{\partial}{\partial z}(xy)\,\hat{z}$$

$$= y\hat{x} + x\hat{y}$$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = \hat{x}\,0 + \hat{y}\left(+\frac{\partial x}{\partial z}\right) + \hat{z}\left(-\frac{\partial x}{\partial y}\right) = 0$$

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = \hat{x}\,0 + \hat{y}\,\frac{\partial y}{\partial z} - \hat{z}\,\frac{\partial y}{\partial y} = -\hat{z}$$

$$\vec{A}\times(\vec{\nabla}\times\vec{B}) = x\hat{x}\times(-\hat{z}) = x\hat{y}$$

$$(\vec{A}\cdot\vec{\nabla})\vec{B} = x\frac{\partial}{\partial x}\vec{B} = 0$$

$$(\vec{B}\cdot\vec{\nabla})\vec{A} = y\frac{\partial}{\partial x}\vec{A} = y\hat{x}$$

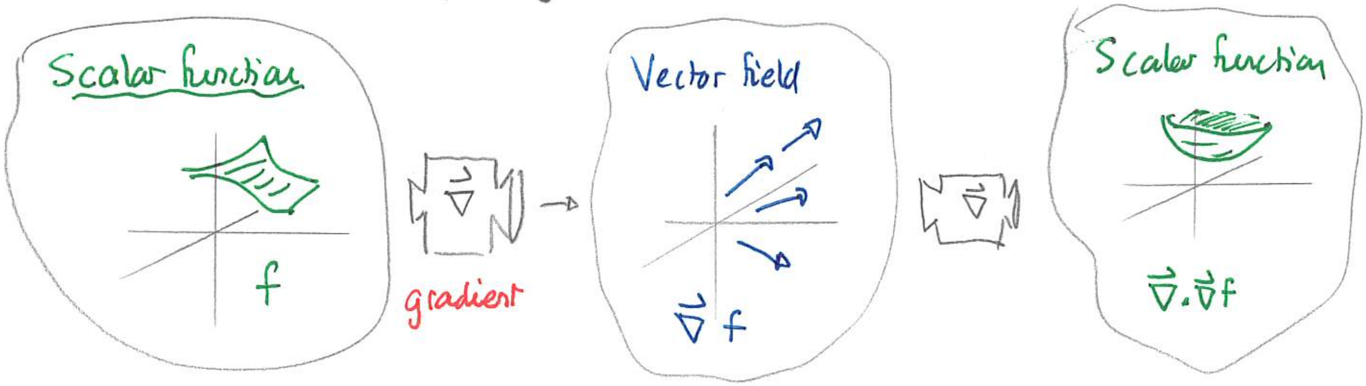
So $\vec{\nabla}(\vec{A}\cdot\vec{B}) = A\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$

b) $\vec{\nabla}\cdot\vec{B} = 0$ and $(\vec{\nabla}\cdot\vec{B})\vec{A} = 0$

But $(\vec{B}\cdot\vec{\nabla})\vec{A} = y\hat{x}$ $\longrightarrow$ NOT EQUAL

Multiple derivatives

One can differentiate repeatedly. For example



Here

$$\vec{\nabla}\cdot\vec{\nabla} f = \vec{\nabla}\cdot\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

We define the <u>Laplacian operator</u> $\vec{\nabla}^2$ via

$$\boxed{f \xrightarrow{\vec{\nabla}^2} \vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

In electromagnetism we will eventually have

$$\vec{\nabla}^2 V = \text{function of charge.}$$

Certain multiple derivatives satisfy general rules. For example,

$$\boxed{\begin{array}{l}\text{For any suitable differentiable function } f \\ \vec{\nabla} \times \vec{\nabla} f = 0\end{array}}$$

The intuitive understanding of this is that $\vec{\nabla} \times \vec{V}$ captures the extent to which $\vec{V}$ circles. So $\vec{\nabla} \times \vec{\nabla} f$ captures the extent to which the gradient circles. But the gradient "points uphill" and so it cannot circle.

We can also show

$$\boxed{\begin{array}{l}\text{For any suitably differentiable vector field, } \vec{v} \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0\end{array}}$$

Integration in three dimensions

We will encounter

* integrals of scalar functions
* integrals of vector functions - line or surface integrals.

We will develop methods to evaluate these, often using the fundamental theorem of calculus:

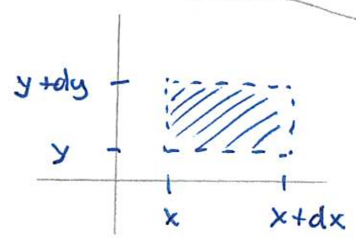
$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

Volume integrals of scalar functions

An example of a scalar function in electromagnetic theory is charge density, which describes the spatial distribution of charge.

The volume charge density $\rho(\vec{r}) = \rho(x,y,z)$ is a scalar function with units C/m^3 whose meaning approximately is: The charge in the region

$x \to x+dx$
$y \to y+dy$
$z \to z+dz$

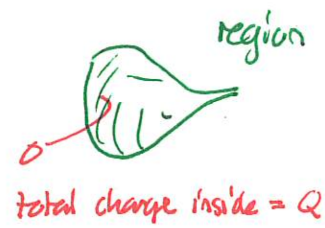


is $\underbrace{\rho(x,y,z)}_{\text{density}}\ \underbrace{dx\,dy\,dz}_{\text{volume}}$ as $dx, dy, dz \to 0$

More precisely

Consider any region of space. Then the charge contained in this region is

$$Q = \iiint_{\text{region}} \rho(x,y,z)\,dx\,dy\,dz$$



We often abbreviate this using

$$d\tau \equiv dx\,dy\,dz$$

and

$$Q = \int \rho \, d\tau$$

Note that this

* Does <u>NOT</u> mean integrate with respect to <u>one variable</u> τ.

* Does NOT regard ρ as constant

* It <u>DOES</u> mean
  - write $\rho = \rho(x,y,z)$
  - write $d\tau = dx\,dy\,dz$
  - integrate w.r.t. x, y, z

4 Three dimensional charge distribution

Within the region $-a \leqslant x, y, z \leqslant a$ with $a > 0$ the charge density is

$$\rho(x, y, z) = \frac{4q}{a^5}\, xy$$

where q is a constant with dimensions of charge.

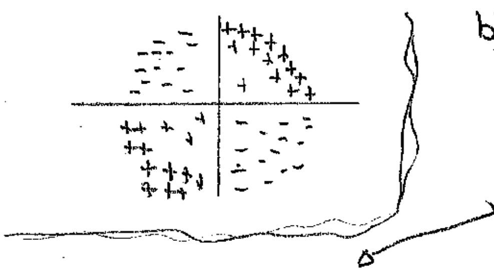
a) In the region $-a \leqslant xy \leqslant a$ in the xy plane sketch whether the charge is positive or negative indicating regions with greater and smaller charge density.

b) Determine the total charge in the region $-a \leqslant x, y, z \leqslant a$.

c) Determine the total charge in the region $0 \leqslant x, y, z \leqslant a$.

d) Determine the total charge contained in the segment of a sphere of radius a in the quadrant where $0 \leqslant x, y, z \leqslant a$.

Answers: a)



b) In this region

$$\left.\begin{aligned} -a &\leq x \leq a \\ -a &\leq y \leq a \\ -a &\leq z \leq a \end{aligned}\right\} d\tau = dx\,dy\,dz$$

b) $$Q = \int \rho\, d\tau = \int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} dz\, \frac{4q}{a^5} xy = \frac{4q}{a^5} \int_{-a}^{a} x\,dx \int_{-a}^{a} y\,dy \int_{-a}^{a} dz$$

$$= \frac{4q}{a^5} \underbrace{\left.\frac{x^2}{2}\right|_{-a}^{a}}_{0} \underbrace{\left.\frac{y^2}{2}\right|_{-a}^{a}}_{0} \left. x \right|_{-a}^{a}$$

$\Rightarrow Q = 0$

c) Here

$$\begin{aligned} 0 &\leq x \leq a \\ 0 &\leq y \leq a \\ 0 &\leq z \leq a \end{aligned}$$

$$Q = \int_0^a dx \int_0^a dy \int_0^a dz\, \frac{4q}{a^5} xy = \frac{4q}{a^5} \underbrace{\left.\frac{x^2}{2}\right|_0^a}_{a^2/2} \underbrace{\left.\frac{y^2}{2}\right|_0^a}_{a^2/2} \underbrace{\left. z \right|_0^a}_{a}$$

$\Rightarrow Q = q$