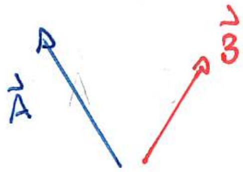
Fri: HW1 by 5pm

Tues: HW2 by 5pm

Read 1.2.4, 1.2.5, 1.2.6.

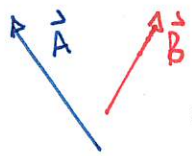
Vector Triple Products USEFUL but TRICKY!

Recall that the dot product maps two vectors to a scalar



$\xrightarrow[\text{dot prod.}]{}$ $\vec{A}\cdot\vec{B}$ (number)

The cross product maps two vectors to a vector



$\xrightarrow[\text{cross prod}]{}$ $\vec{A}\times\vec{B}$ (vector)

We can repeatedly make products. Examples are

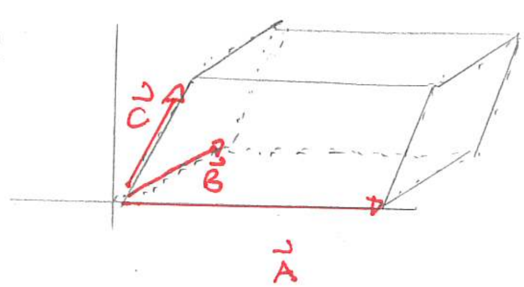
1) Scalar triple product

Consider three vectors $\vec{A}, \vec{B}, \vec{C}$. Then $\vec{A}\cdot(\vec{B}\times\vec{C})$ is a scalar. One can prove

a) $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

b) The volume of a parallel sided box made by $\vec{A}, \vec{B}, \vec{C}$ is

$\vec{A}\cdot(\vec{B}\times\vec{C})$



2) <u>Vector triple product</u>

This maps three vectors $\vec{A}, \vec{B}, \vec{C}$ using the cross product twice to produce. Note

a) In general $\underbrace{\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}}_{\text{Beware.}}$. There can be exceptions e.g. $\vec{A} = \vec{B} = \vec{C}$

b) $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ "BAC CAB" rule

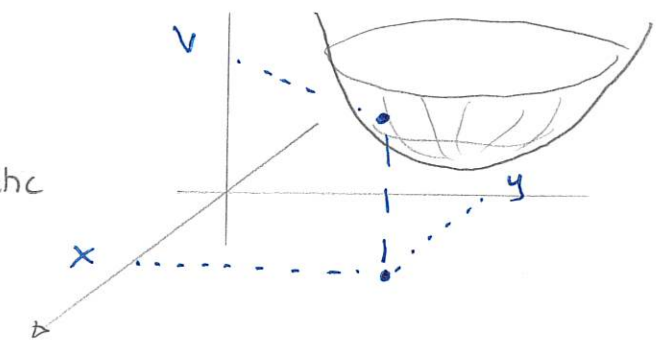
Functions of position

Electromagnetic theory consider various quantities that vary depending on location. These are therefore functions of position. Examples include

1) charge density (charge per unit volume) — one number at each location
2) electrostatic potential (Volts) —
3) electric field — one vector at each location
4) magnetic field —

The two basic types of functions that we consider are:

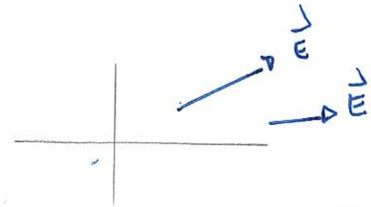
1) scalar functions A scalar function returns a single number at each location. For example, the electrostatic potential due to a point charge at the origin maps



location

$$\underbrace{\vec{r} \equiv (x,y,z)}_{\text{location}} \xrightarrow[\text{function}]{\text{potential}} V(\vec{r}) \equiv V(\underbrace{x,y,z}_{\text{provide location}}) = \overbrace{\underbrace{\frac{1}{4\pi\epsilon_0}\frac{q}{\left[x^2+y^2+z^2\right]^{1/2}}}_{\text{gives potential}}}^{\text{function output}}$$

2) vector functions A vector function returns a single vector at each location. For example, the electric field due to a point charge at the origin



$$\overset{\text{location}}{\vec{r} = (x,y,z)} \xrightarrow[\text{function}]{\text{electric field}} \vec{E} = \vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{q}{(x^2+y^2+z^2)^{3/2}}\overbrace{(x\hat{x}+y\hat{y}+z\hat{z})}^{\text{vector}}$$

Vector functions are often called <u>vector fields</u>

We now develop differentiation procedures for both types of functions

Differentiation of scalar functions

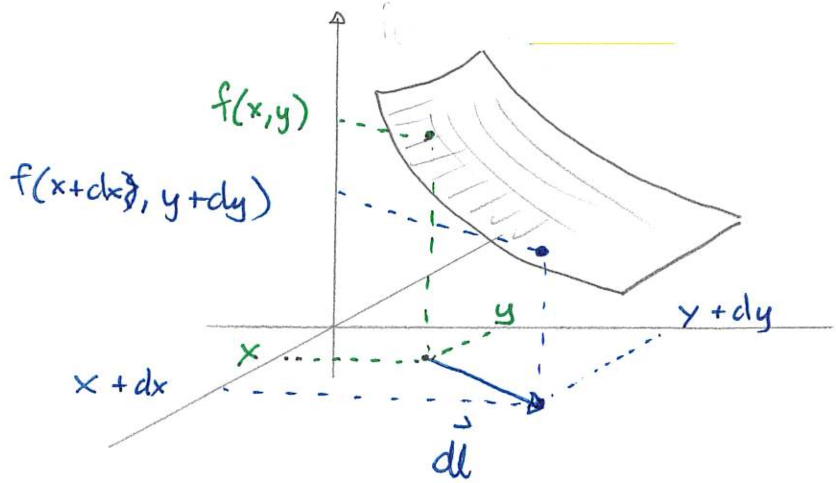
The basic idea of differentiation to the determine the rate at which a function varies as its arguments vary. For functions of location, there are many ways in which the three argument variables can vary independently. We will provide a differentiation system that encapsulates all possiblities. The scheme is:

Choose an "initial" location Cartesion co-ords.

$\vec{r} = (x, y, z)$

and evaluate

$f(x, y, z)$



Choose a "final" location

$\vec{r} + \vec{dl} = (x+dx, y+dy, z+dz)$

and evaluate

$f(x+dx, y+dy, z+dz)$

$\rightarrow$ Multivariable calculus

$$f(x+dx, y+dy, z+dz) \simeq f(x,y,z) + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

evaluate at x, y, z

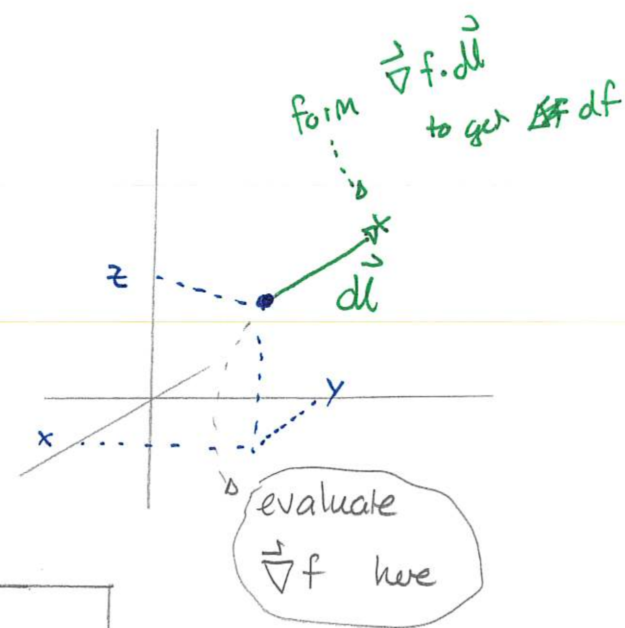
Then the change in f is

$$\boxed{df = f(x+dx, y+dy, z+dz) - f(x,y,z) \simeq \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz}$$

This prepares us to evaluate the change for all possible changes in location, which are described by

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

where dx, dy and dz can vary independently. The crucial ingredient that allows us to evaluate the change in f is the collection $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \ldots$ We thus define



The gradient of $f(x,y,z)$ is the vector $$\vec{\nabla} f := \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$ provided that x, y, z are Cartesian co-ordinates

Then for infinitesimal $\vec{dl}$

$$df = f(\vec{r} + \vec{dl}) - f(\vec{r})$$
$$= \vec{\nabla} f \cdot \vec{dl}$$

or

$$df = \vec{\nabla} f \cdot \vec{dl}$$

1 Scalar function differentiation

Consider

$$f(\mathbf{r}) = f(x, y, z) = \frac{6}{y}.$$

~~where $a > 0$.~~

a) Determine the change, $\mathrm{d}f$, given that the initial location is $(2, 2, 0)$ and $\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}}$.

b) Determine the change, $\mathrm{d}f$, given that the initial location is $(2, 2, 0)$ and $\mathrm{d}\mathbf{l} = \mathrm{d}y\,\hat{\mathbf{y}}$.

In both cases,

$$\vec{\nabla} f = \underbrace{\cancel{\frac{\partial f}{\partial x}}}_{0}\hat{x} + \underbrace{\frac{\partial f}{\partial y}}_{-6/y^2}\hat{y} + \overbrace{\cancel{\frac{\partial f}{\partial z}}}^{0}\hat{z}$$

$$= -\frac{6}{y^2}\hat{y}$$

and at $y = 2$

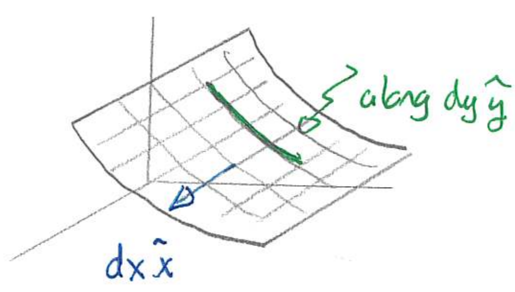
$$\vec{\nabla} f = -\frac{6}{4}\hat{y} = -\frac{3}{2}\hat{y}$$

a) $df = \vec{\nabla} f \cdot \vec{dl} = -\frac{3}{2}\hat{y} \cdot dx\,\hat{x} = -\frac{3}{2}dx\,\cancel{\hat{y}\cdot\hat{x}} = 0$

b) $df = \vec{\nabla} f \cdot \vec{dl} = -\frac{3}{2}\hat{y} \cdot dy\,\hat{y} = -\frac{3}{2}dy$

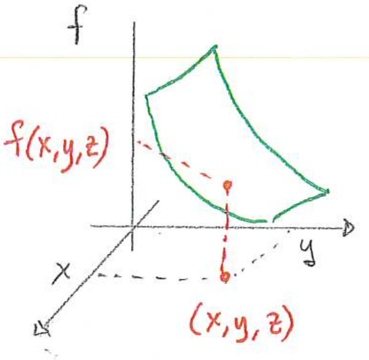
Observing the function we see that along the x-direction it is constant. So we expect $df = 0$.

Along the y-direction it decreases. So for $dy > 0$ we expect $df < 0$



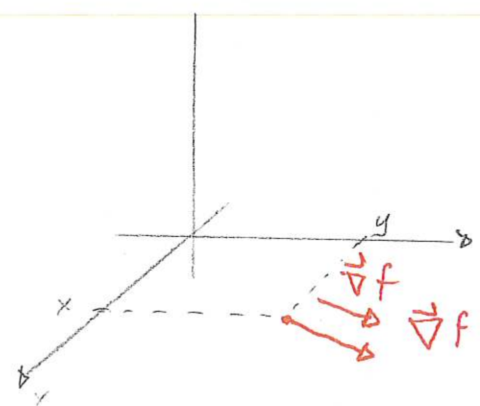
The conceptual scheme is

Given a function f we want changes df all starting at (x,y,z)



e.g. $f = \dfrac{1}{x^2+y^2}$

The gradient provides a vector that captures all possible changes. Calculate the vector $\vec{\nabla} f$ at each point

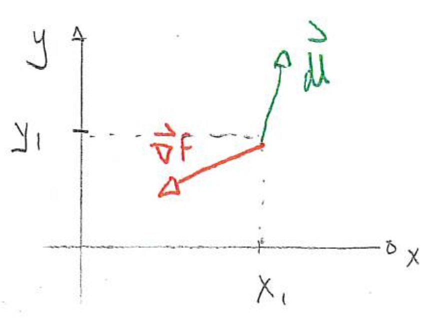


e.g. $\vec{\nabla} f = \dfrac{-2x}{(x^2+y^2)^2}\hat{x} - \dfrac{2y}{(x^2+y^2)^2}\hat{y}$

To find an actual change in f:

* describe location (x_1, y_1, z_1)
* evaluate $\vec{\nabla} f$ at (x_1, y_1, z_1)
* get direction of change of positions

$$\vec{dl} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$



The change in f is

$$df \approx \vec{\nabla} f \cdot \vec{dl}$$

e.g. $\vec{dl} = dx\,\hat{x} + dy\,\hat{y}$

$$df \approx \frac{-2x\,dx - 2y\,dy}{(x^2+y^2)^2}$$

With these definitions one can prove:

1) <u>direction of gradient</u>

The gradient of f, $\vec{\nabla} f$, is perpendicular to contours along which f is constant. It points in direction of increasing f.

To show this, note that along a contour $df = 0$. Let $\vec{dl}$ be tangent to the contour. Then

$$df = \vec{\nabla} f \cdot \vec{dl} = 0.$$

Thus $\vec{\nabla} f$ is perpendicular. Now

$$df = \vec{\nabla} f \cdot \vec{dl} = |\vec{\nabla} f||dl| \cos\theta$$

and when $\theta = 0$ $df > 0$ $\Rightarrow$ increases

2) <u>maximum rate of change</u>

The maximum rate of change of f is along ∇f and has magnitude $|\vec{\nabla} f|$

Here

$$df = |\vec{\nabla} f||dl| \cos\theta$$

$$\Rightarrow \frac{df}{|\vec{dl}|} = |\vec{\nabla} f| \cos\theta$$

To maximize $\theta = 0$ and the outcome is $|\vec{\nabla} f|$

2 Gradients

Let

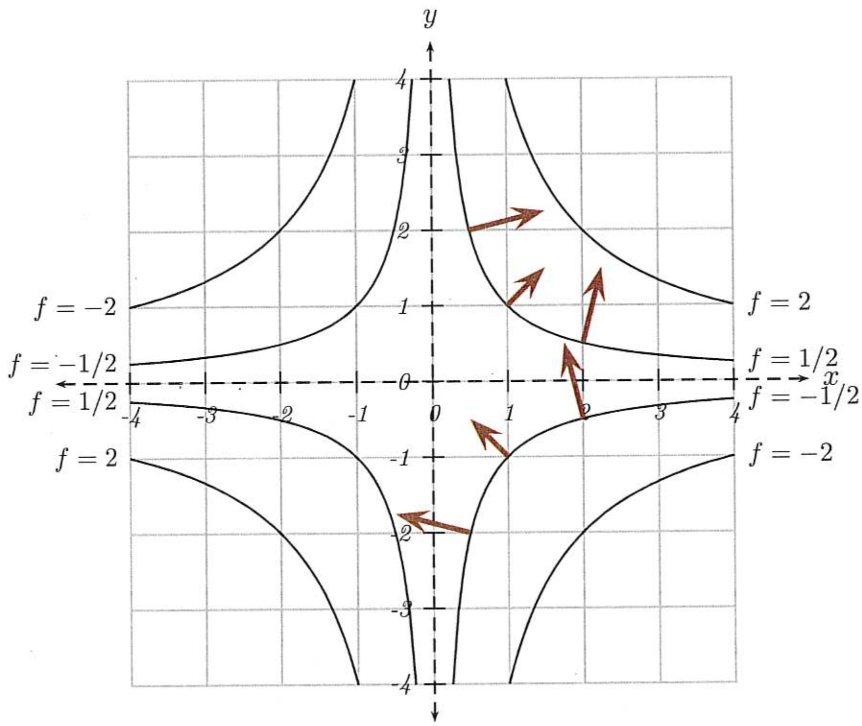
$$f(x, y) = \frac{1}{2} xy$$

be a function in two dimensions.

a) Sketch the contours of f for which $f(x, y) = 1/2$, $f(x, y) = 2$, $f(x, y) = -1/2$, and $f(x, y) = -2$. Indicate the directions in which $f(x, y)$ increases and decreases in all four quadrants.

b) Determine ∇f. Sketch the resulting vectors along the $f(x, y) = 1/2$ contour at points such as $(1, 1)$, $(2, 1/2)$, ... Repeat this for the $f(x, y) = -1/2$ contour. Are these consistent with the directions in which the function increases or decreases?

Answer:



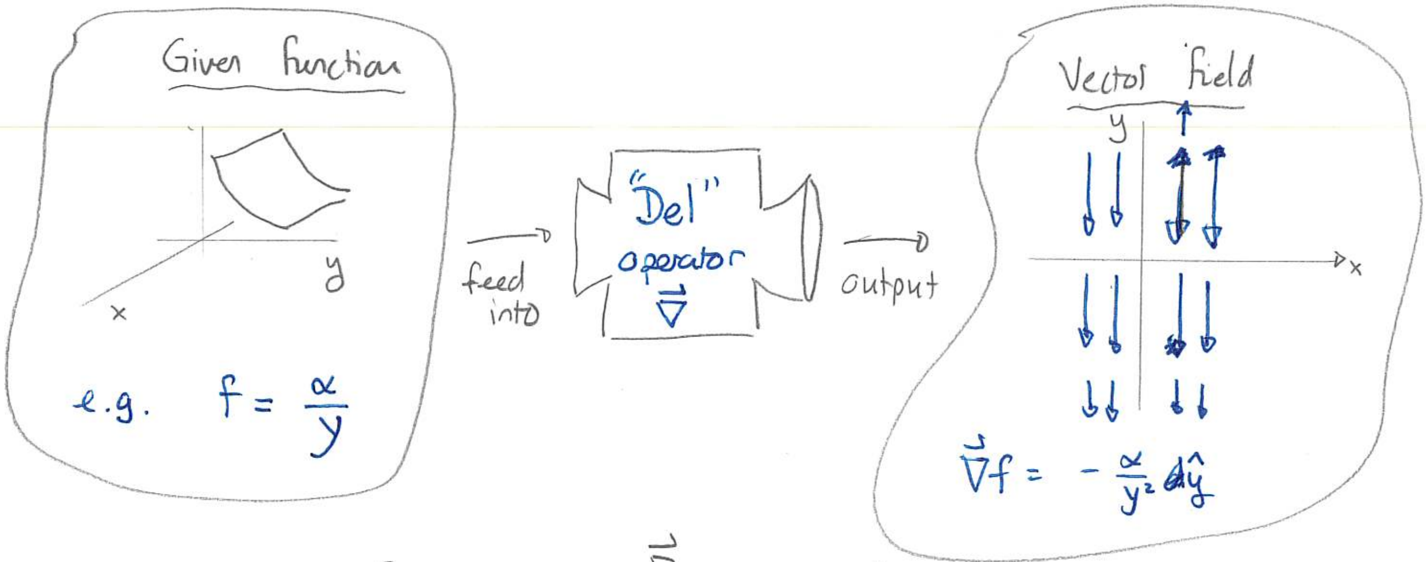
b) $\vec{\nabla} f = \frac{1}{2} y \hat{x} + \frac{1}{2} x \hat{y}$

At $(x, y) = (1, 1)$ $\vec{\nabla} f = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y}$ this's is illustrated

etc....

"Del" operator

The process of form the gradient is an operation and we can associate an operator with this process Schematically,



$$f(x,y,z) \xrightarrow{\vec{\nabla}} \vec{\nabla} f(x,y,z)$$

The operator can be regarded as existing independently of the input. So

$$\vec{\nabla}(\ldots) = \hat{x}\frac{\partial}{\partial x}(\ldots) + \hat{y}\frac{\partial}{\partial y}(\ldots) + \hat{z}\frac{\partial}{\partial z}(\ldots)$$

insert same function into all slots

Thus we define the "del" operator as:

$$\boxed{\vec{\nabla} := \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.}$$

Note: 1) $\vec{\nabla}$ takes a function as an input.

2) The derivatives do not act on basis vectors $\frac{\partial}{\partial x} x = 1$ and $\frac{\partial}{\partial x}\hat{x} \neq 1$

3) Always include basis vectors when using $\vec{\nabla}$

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \neq \underbrace{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}}_{\text{scalar}}$$

vectors ESSENTIAL

Differentiating vector fields

In general a vector field can vary from one location to another and we would like to determine rates of change. In general a vector field has form

$$\vec{V} \equiv \vec{V}(\vec{r}) \equiv \vec{V}(\underbrace{x,y,z}_{\text{describes location}}) = \underbrace{V_x}_{\text{labels which component}}(\underbrace{x,y,z}_{\text{location}})\,\underbrace{\hat{x}}_{\text{basis vector}} + V_y(x,y,z)\,\hat{y} + V_z(x,y,z)\,\hat{z}$$

When differentiating a vector field we require:

> Differentiation must give the same results and different Cartesian co-ordinate systems.

We can provide an example of a bad definition of a vector derivative

Example (BAD DEFINITION)

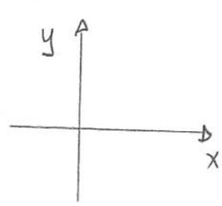
Suppose we construct

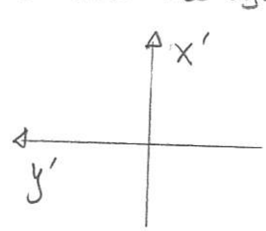
$$\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z}$$

in one co-ordinate system. Then, in another Cartesian co-ordinate system with co-ordinates x', y', z' we would form

$$\frac{\partial V_x'}{\partial x'}\hat{x}' + \frac{\partial V_y'}{\partial y'}\hat{y}' + \frac{\partial V_z'}{\partial z'}\hat{z}'$$

where V_x', V_y', V_z' are vector components in this system. These two vectors should match. As an example consider the systems





Then $x' = y$

$y' = -x$

$\hat{x}' = \hat{y}$

$\hat{y}' = -\hat{x}$

Suppose that $V_x = x$, $V_y = V_z = 0$ in the unprimed system.

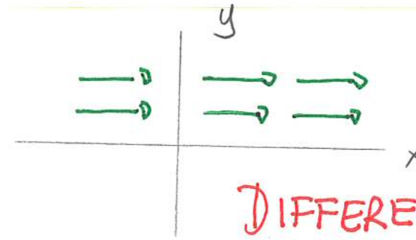
Then

$$\vec{V} = V_x \hat{x} = x\hat{x} = -y'(-\hat{y}')$$

$$= y'\hat{y}'$$

Now in the unprimed system

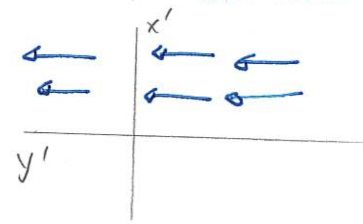
$$\frac{\partial V_x}{\partial x}\hat{x} + \frac{\partial V_y}{\partial y}\hat{y} + \frac{\partial V_z}{\partial z}\hat{z} = \hat{x}$$



DIFFERENT!

Then in the primed system

$$\frac{\partial V_x'}{\partial x'}\hat{x}' + \frac{\partial V_y'}{\partial y'}\hat{y}' + \frac{\partial V_z'}{\partial z'}\hat{z}' = \hat{y}' = -\hat{x}$$



The two procedures give different results. The definition is not good.

END (BAD DEFINITION) ∎