Lecture 1

Handout * Syllabus

Intro * Course Website + materials page

* Assignments - usually due Friday, Tuesday
    - posted to course website
    - first Friday, Aug 21

* Exams - two in class.
    - final in usual final period.

Thurs: Read 1.1.3, 1.2.1, 1.2.2, 1.2.3. HW: Due Friday

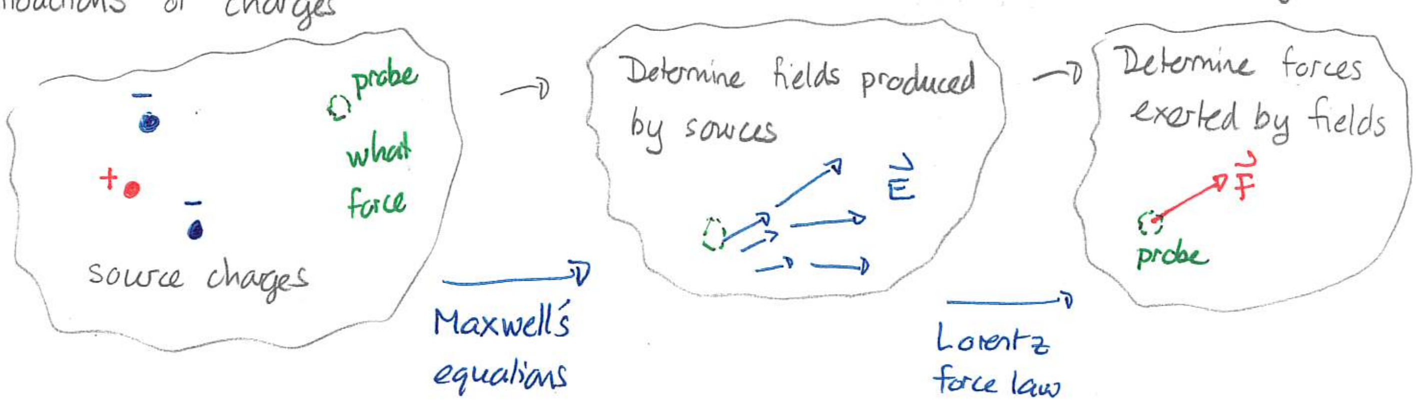
Electromagnetism

Electromagnetism considers how charged particles interact. It offers methods for determining the forces exerted using intermediate entities called electric + magnetic fields.

DEMO: PhET Radio Waves + Electromagnetic Fields

* Hide fields + show electron motion
* Show fields

Electromagnetic theory provides a complete and compact formalism for describing the interactions of charges



Electromagnetic theory is a core part of physics and is important because

1) it is essential for describing the physics of situations involving charged particles, e.g.
   - atomic structure
   - trapped ions DEMO: Oxford Page Ion Trap.
   - radio wave production + detection
2) it is useful for describing the interaction between light + matter.

This Course

Phys 311 is a first course in electromagnetic theory, starting from the basic principles of electromagnetism. It aims to:

1) provide a complete description of electric + magnetic fields
2) provide basic techniques for determining electric + magnetic fields in all situations
3) arrive at Maxwell's equations.

The course will use + develope.

1) basic classical mechanics - Newton's Laws, energy..
2) vector algebra, vector calculus
3) calculus in three dimensions

Vector algebra 1.1.1

Electromagnetic theory uses vectors to describe:

* locations of charges + currents
* directions of currents
* electric + magnetic fields.

These eventually require

1) vector algebra ⇝ algebraic operations between vectors (add, multiply, ...)
2) vector calculus ⇝ differentiation + integration of vector functions.

The basic conceptual ideas can be illustrated using pictorial manipulations of displacement vectors. These can eventually be abstracted to give a notion of a vector space:

See Halmos "Finite Dimensional Vector Spaces"

A vector space is a set of objects with two operations

1) addition ⇒ for any vectors $\vec{A}, \vec{B}$ the sum $\vec{A}+\vec{B}$ is also a vector
2) scalar multiplication ⇒ for any vector $\vec{A}$ and scalar λ, $\lambda\vec{A}$ is a vector.

These operations must satisfy:

a) addition is commutative $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

b) " " associative $\vec{A}+(\vec{B}+\vec{C}) = (\vec{A}+\vec{B})+\vec{C}$

c) there is a zero vector $\vec{0}$ s.t. $\vec{A}+\vec{0} = \vec{A}$

d) for any vector $\vec{A}$, there is an additive inverse $(-\vec{A})$ s.t. $\vec{A}+(-\vec{A}) = \vec{0}$

e) for any vector $\vec{A}$ and scalars α, β $\alpha(\beta\vec{A}) = (\alpha\beta)\vec{A}$

f) for the scalar 1 and any $\vec{A}$, $1\vec{A} = \vec{A}$

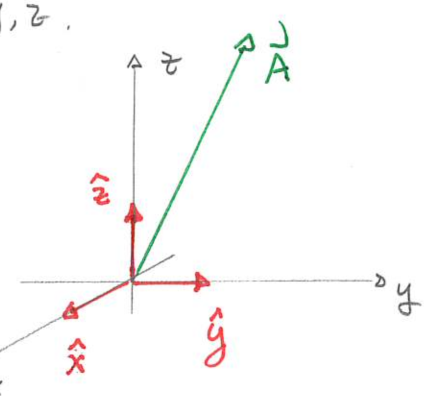
g) for any vectors $\vec{A}, \vec{B}$ and scalar α $\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha\vec{B}$

h) " " vector $\vec{A}$ and scalars α, β $(\alpha+\beta)\vec{A} = \alpha\vec{A}+\beta\vec{A}$

There are many examples of vector spaces. Linear algebra provides the mathematical theory for these.

Vector bases + components 1.1.2

A displacement vector in three dimensions, $\vec{A}$, can be expressed in terms of three unit vectors directed along Cartesian axes, x, y, z.



vector

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

scalar

where A_x, A_y and A_z are three scalars that are unique to the vector $\vec{A}$. These are called the components of $\vec{A}$. Various theorems of linear algebra guarantee the existence and uniqueness of these scalars for a given basis. We can then represent the vector in terms of the three scalars. For example

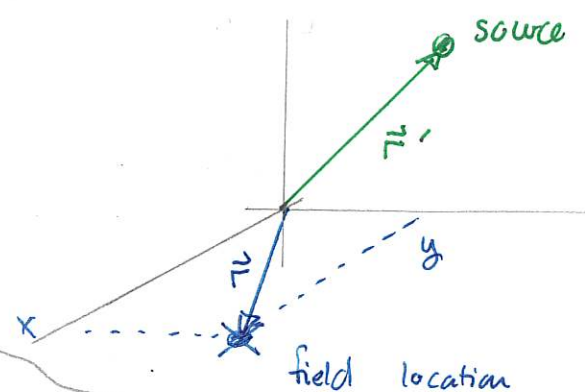
$$\vec{A} \leftrightarrow \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad \text{or} \quad \vec{A} \leftrightarrow \langle A_x, A_y, A_z \rangle.$$

and define addition and scalar multiplication of these using the components.

Position vectors 1.1.4.

In electromagnetic theory we will use position vectors to describe

1) location of source charges
2) locations at which fields are to be calculated.



Specifically

(unprimed) $\vec{r}$ ≡ position vector for field location
= displacement vector from origin to field point

(primed) $\vec{r}\,'$ = position vector for source location

= displacement vector from origin to source location

The components of these position vectors are the co-ordinates of the points. Thus

Position vector for field point at (x, y, z) is

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

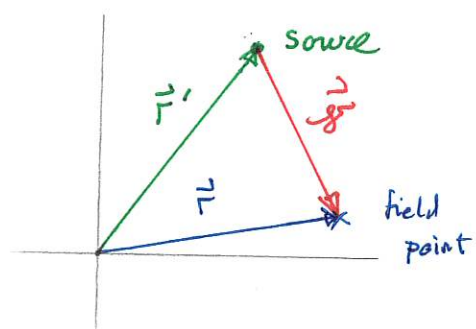
and

Position vector for source at (x', y', z') is

$$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Separation vectors

In electromagnetic theory the location of a field point relative to a source charge will be vital. We then define the separation vector as the displacement from the source to the field point. Thus



Given source point $\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$

field point $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

the separation vector is $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

$$= (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}$$

1 Pyramid

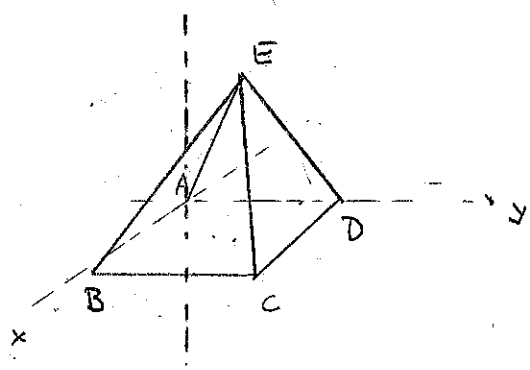
A pyramid has a square base with sides of length L. The apex of the pyramid is a height h above the center of the base. The base lies in the first quadrant of the xy plane. Let A be the corner at the origin, B that along the x axis, C that away from either axis and D that along the y axis. Let E be the apex.

a) Determine expressions in terms of the standard basis vectors for the position vector of each corner.

b) Determine an expressions in terms of the standard basis vectors for the separation vector from B to C.

c) Determine an expressions in terms of the standard basis vectors for the separation vector from B to E.

d) Determine an expressions in terms of the standard basis vectors for the separation vector from C to E.



a) $\vec{r}_A = 0\hat{x} + 0\hat{y} + 0\hat{z} = \vec{0}$

$\vec{r}_B = L\hat{x} + 0\hat{y} + 0\hat{z} = L\hat{x}$

$\vec{r}_C = L\hat{x} + L\hat{y} + 0\hat{z} = L\hat{x} + L\hat{y}$

$\vec{r}_D = 0\hat{x} + L\hat{y} + 0\hat{z} = L\hat{y}$

$\vec{r}_E = \frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}$

b) $\vec{\mathscr{r}}_{B\ \text{to}\ C} = \vec{r}_C - \vec{r}_B = L\hat{x} + L\hat{y} - L\hat{x} \Rightarrow \vec{\mathscr{r}}_{B\ \text{to}\ C} = L\hat{y}$

c) $\vec{\mathscr{r}}_{B\ \text{to}\ E} = \vec{r}_E - \vec{r}_B = \frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z} - L\hat{x} \Rightarrow \vec{\mathscr{r}}_{B\ \text{to}\ E} = -\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}$

d) $\vec{\mathscr{r}}_{C\ \text{to}\ E} = \vec{r}_E - \vec{r}_C = \frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z} - (L\hat{x} + L\hat{y})$

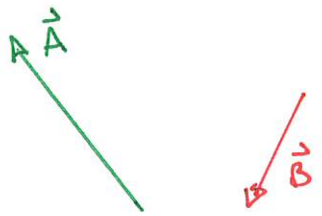
$\Rightarrow \vec{\mathscr{r}}_{C\ \text{to}\ E} = -\frac{L}{2}\hat{x} - \frac{L}{2}\hat{y} + h\hat{z}$

Dot product 1.1.1.

Vector spaces can be given additional operations besides vector addition and scalar multiplication. An example is the dot product, a form of inner product. The dot product performs as follows:

Input: two vectors

Output: Single number.



$\longrightarrow$

Denoted

$\vec{A}\cdot\vec{B} \equiv$ number

In Cartesian co-ords:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

This can easily be shown to satisfy the abstract properties:

1) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

2) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

3) $(\lambda\vec{A})\cdot\vec{B} = \lambda(\vec{A}\cdot\vec{B})$

4) $\vec{A}\cdot\vec{A} \geqslant 0$ and $\vec{A}\cdot\vec{A} = 0 \Leftrightarrow \vec{A} = 0$

The product can be computed using these rules and the special rules for Cartesian basis vectors:

$\hat{x}\cdot\hat{x} = 1$ $\quad$ $\hat{x}\cdot\hat{y} = 0$

$\hat{y}\cdot\hat{y} = 1$ $\quad$ $\hat{x}\cdot\hat{z} = 0$

$\hat{z}\cdot\hat{z} = 1$ $\quad$ $\hat{y}\cdot\hat{z} = 0$ $\quad$ etc...

The dot product is connected to geometrical features of vectors.

For any vector $\vec{A}$, the magnitude of the vector is

$$A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

Then the Cauchy-Schwarz inequality gives

$$-1 \leq \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} \leq 1$$

and so we can define the angle between two vectors by:

If $\vec{A}, \vec{B}$ are two vectors then the angle, θ, between them satisfies

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

Thus

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta.$$

Separately

$\vec{A}$ and $\vec{B}$ are orthogonal $\Leftrightarrow$ the angle between them is 90°

$\Leftrightarrow \vec{A}\cdot\vec{B} = 0$

2 Pyramid face angles

A pyramid has a square base with sides of length L. The apex of the pyramid is a height h above the center of the base. The base lies in the first quadrant of the xy plane. Let A be the corner at the origin, B that along the x axis, C that away from either axis and D that along the y axis. Let E be the apex.

a) Determine an expression for the length of the side from B to E.

b) Determine the angle on the EBC face at point B.

c) Determine the angle on the EBC face at point E.

Answer: a) $\vec{r}_{B\text{ to }E}$ is the vector along the side

$$r_{B\text{ to }E} = \sqrt{\vec{r}_{B\text{ to }E} \cdot \vec{r}_{B\text{ to }E}}$$

$$= \left[\left(-\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}\right)\cdot\left(-\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}\right)\right]^{1/2}$$

$$= \sqrt{\frac{L^2}{4} + \frac{L^2}{4} + h^2} \quad \Rightarrow \quad r_{B\text{ to }E} = \sqrt{L^2/2 + h^2}$$

b) We need

$$\cos\theta_B = \frac{\vec{r}_{B\text{ to }E} \cdot \vec{r}_{B\text{ to }C}}{|r_{B\text{ to }E}|\,|r_{B\text{ to }C}|}$$

Note $\vec{r}_{B\text{ to }C} = L\hat{y} \quad \Rightarrow \quad r_{B\text{ to }C} = L$

Then

$$\vec{r}_{B\text{ to }E} \cdot \vec{r}_{B\text{ to }C} = \left(-\frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} + h\hat{z}\right)\cdot L\hat{y}$$

$$= -\frac{L^2}{2}\cancelto{0}{\hat{x}\cdot\hat{y}} + L^2/2\,\cancelto{1}{\hat{y}\cdot\hat{y}} + hL\,\cancelto{0}{\hat{z}\cdot\hat{y}} = L^2/2$$

$$\Rightarrow \quad \cos\theta = \frac{L^2/2}{\sqrt{L^2/2 + h^2}\; L}$$

Thus

$$\cos\theta = \frac{L}{2}\frac{1}{\sqrt{L^2/2 + h^2}}$$

$$= \frac{L}{2}\frac{1}{\left[L^2\left(\frac{1}{2}+h^2/L^2\right)\right]^{1/2}} = \frac{1}{2\left(1/2 + h^2/L^2\right)^{1/2}}$$

Note that as $h \to 0 \quad \cos\theta \to \frac{1}{\sqrt{2}} \Rightarrow \theta \to \pi/4$

$h \to \infty \quad \cos\theta \to 0 \Rightarrow \theta \to \pi/2$

c) This is the angle between $\vec{r}_{E\text{ to }B} = -\vec{r}_{B\text{ to }E} = \frac{L}{2}\hat{x} - \frac{L}{2}\hat{y} - h\hat{z}$

$\vec{r}_{E\text{ to }C} = -\vec{r}_{C\text{ to }E} = \frac{L}{2}\hat{x} + \frac{L}{2}\hat{y} - h\hat{z}$

Then $r_{E\text{ to }B} = \sqrt{L^2/2 + h^2}$

$r_{E\text{ to }C} = \sqrt{L^2/2 + h^2}$

$$\vec{r}_{E\text{ to }B} \cdot \vec{r}_{E\text{ to }C} = \frac{L^2}{4} - \frac{L^2}{4} + h^2 = h^2$$

So

$$\cos\theta = \frac{\vec{r}_{E\text{ to }B} \cdot \vec{r}_{E\text{ to }C}}{|\vec{r}_{E\text{ to }B}|\,|\vec{r}_{E\text{ to }C}|} = \frac{h^2}{L^2/2 + h^2}$$

$$\cos\theta = \frac{1}{1 + \frac{L^2}{2h^2}}$$

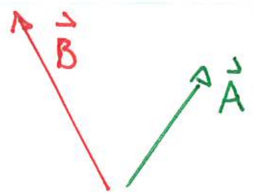
Then as $h \to 0 \quad \cos\theta \to 0 \Rightarrow \theta \to \pi/2$ (two lines intersect in the plane)

$h \to \infty \quad \cos\theta \to 1 \Rightarrow \theta \to 0$ (lines are parallel.

Vector cross product

The cross product is an antisymmetric form of vector multiplication

Input: Two vectors



Output: single vector

$\odot\ \vec{A}\times\vec{B}$

In Cartesian basis

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

The cross product satisfies

1) $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$
2) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ Not commutative
3) $(\lambda\vec{A})\times\vec{B} = \lambda(\vec{A}\times\vec{B})$
4) $\vec{A}\times\vec{A} = 0$

Then for Cartesian basis vectors

$$\hat{x}\times\hat{x} = 0 \qquad \hat{y}\times\hat{y} = 0 \qquad \hat{z}\times\hat{z} = 0$$
$$\hat{x}\times\hat{y} = \hat{z} \qquad \hat{y}\times\hat{z} = \hat{x} \qquad \hat{z}\times\hat{x} = \hat{y}$$
$$\hat{y}\times\hat{x} = -\hat{z} \qquad \hat{z}\times\hat{y} = -\hat{x} \qquad \hat{x}\times\hat{z} = -\hat{y}$$

3 Cross product algebra

The cross product satisfies

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1a}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1b}$$

$$(\lambda \mathbf{A}) \times \mathbf{B} = \lambda \mathbf{A} \times \mathbf{B} \tag{1c}$$

with similar rules when the factors are interchanged. Using only the rules of Eqs. (1) and those for the cross products of basis vectors determine $\mathbf{A} \times \mathbf{B}$ for

$$\mathbf{A} = \hat{\mathbf{x}} + 4\hat{\mathbf{y}} + 9\hat{\mathbf{z}} \quad \text{and}$$
$$\mathbf{B} = 3\hat{\mathbf{x}} + 2\hat{\mathbf{y}}.$$

Answer:

$$\vec{A} \times \vec{B} = (\hat{x} + 4\hat{y} + 9\hat{z}) \times (3\hat{x} + 2\hat{y})$$

$$= \hat{x} \times 3\hat{x} + \hat{x} \times 2\hat{y}$$
$$+ 4\hat{y} \times 3\hat{x} + 4\hat{y} \times 2\hat{y}$$
$$+ 9\hat{z} \times 3\hat{x} + 9\hat{z} \times 2\hat{y}$$

$$= 3\cancel{\hat{x} \times \hat{x}}^{\,0} + 2\underbrace{\hat{x} \times \hat{y}}_{\hat{z}} + 12\underbrace{\hat{y} \times \hat{x}}_{-\hat{z}} + 8\cancel{\hat{y} \times \hat{y}}^{\,0} + 27\underbrace{\hat{z} \times \hat{x}}_{\hat{y}} + 18\underbrace{\hat{z} \times \hat{y}}_{-\hat{x}}$$

$$= -18\hat{x} + 27\hat{y} - 10\hat{z}$$