# Electromagnetic Theory I: Class Exam II

13 November 2025

**Name:** ______________________________ **Total:** /50

## Instructions

- There are 7 questions on 10 pages.
- Show your reasoning and calculations and always explain your answers.

## Physical constants and useful formulae

| | |
|---|---|
| Permittivity of free space | $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$ |
| Charge of an electron | $e = -1.60 \times 10^{-19}\,\mathrm{C}$ |

### Integrals

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

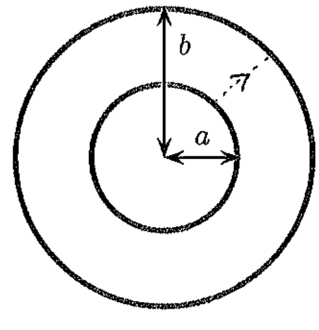
$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

$$\int xe^{-x/a}\,\mathrm{d}x = -a(x+a)e^{-x/a} \qquad \int_0^\infty xe^{-x/a}\,\mathrm{d}x = a^2$$

$$\int x^2e^{-x/a}\,\mathrm{d}x = -a(x^2+2a^2+2ax)e^{-x/a} \qquad \int_0^\infty xe^{-x/a}\,\mathrm{d}x = 2a^3$$

## Question 1

Two hollow spherical conducting spheres are concentric and carry equal and opposite charge. The outer sphere with radius $b$ has charge $-Q$ and the inner sphere with radius $a$ has charge $+Q$. The electric field produced by this is:




$$\mathbf{E} = \begin{cases} 0 & \text{if } r \leqslant a \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} & \text{if } a \leqslant r \leqslant b \\ 0 & \text{if } r \geqslant b. \end{cases}$$

a) Determine the potential difference between the shells and the capacitance of this arrangement.

$\Delta V = -\int \vec{E}\cdot d\vec{l}$ Use a path radially outward. $a \leq r \leq b$

$d\vec{l} = dr\,\hat{r}$

Then $\vec{E}\cdot d\vec{l} = \dfrac{Q}{4\pi\epsilon_0 r^2}dr$

$$\Rightarrow \Delta V = -\int_a^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr = -\frac{Q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} = +\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)\Big|_a^b$$

$\Rightarrow \Delta V = \dfrac{Q}{4\pi\epsilon_0}\left(\dfrac{1}{b} - \dfrac{1}{a}\right)$. The absolute value is $\Delta V = \dfrac{Q}{4\pi\epsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$.

Then $Q = C\,\Delta V \Rightarrow$ $C = \dfrac{1}{4\pi\epsilon_0}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)^{-1}$

b) Suppose that the radius of the outer sphere can be varied. As the radius increases, does the capacitance increase, decrease or stay constant? Explain your answer.

$\frac{1}{b}$ will decrease $\Rightarrow$ $\frac{1}{a} - \frac{1}{b}$ increases $\Rightarrow \left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$ decreases

$\Rightarrow$ decreases

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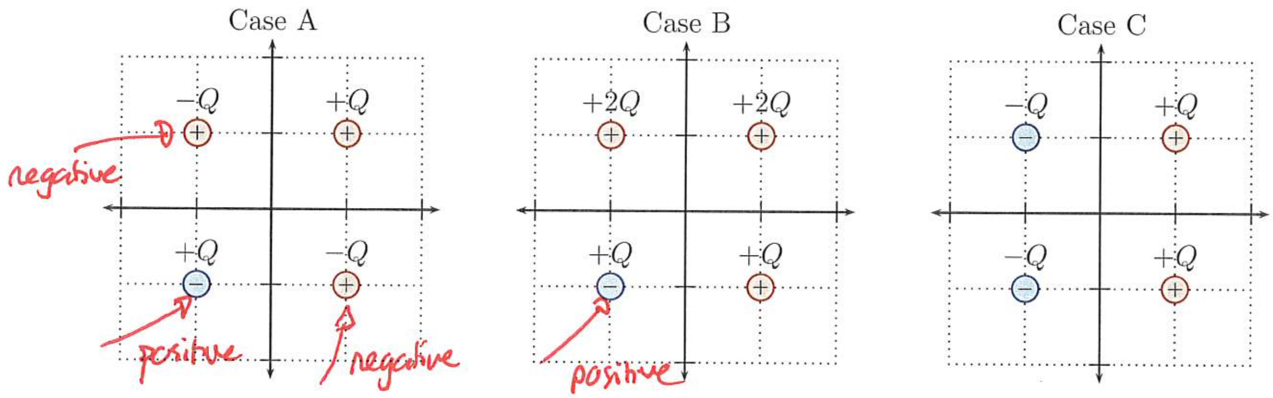
## Question 2

Answer **either part a) or part b) for full credit.** If you answer both, each will be graded and you will be awarded the highest score that you obtained for one of the parts.

a) Consider the illustrated distributions of point charges.



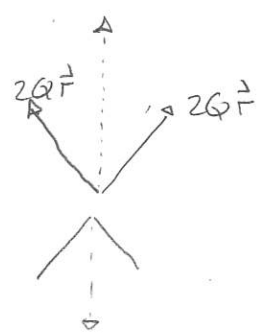

For each distribution describe whether the electric dipole moment is zero or non-zero. If it is non-zero describe its direction. Explain your answer. *Note: You do not have to actually calculate the dipole moment.*

In all cases $\vec{p} = \sum q_i \vec{r}_i$

Case A $q\vec{r}$ cancels for the charges that are diametrically opposite

$\Rightarrow \vec{p} = 0$

Case B The two upper charges give

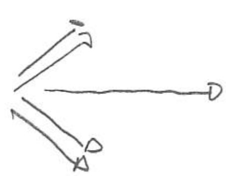



The two lower charges give

Then adding gives

$\vec{p} \neq 0$ $\uparrow$

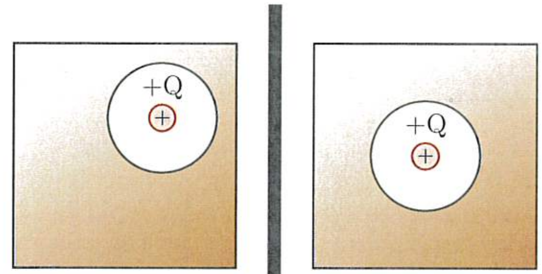
Case C The four are



$\vec{p} \neq 0$ $\rightarrow$

Question 2 continued ...

b) Consider two perfect conductors with spherical cavities. The cavities have the same radius but are centered differently. The same point charge is located at the center of each cavity. The conductors are isolated from each other.




Case A Case B

i) Consider the charge density on the inner surface of each cavity. For each case, are these charge densities uniform or not? Explain your answer.

In all case the conducting surface is an equipotential. Therefore the potential inside the conductor has form the same as a point charge $V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$
where r is the distance from the center of the cavity.
This is the unique solution to $\nabla^2 V = -\rho/\epsilon_0$ inside the cavity. Thus the electric field inside is symmetric. Then $\vec{E}_{conductor} - \vec{E}_{cavity} = \frac{\sigma}{\epsilon_0}\hat{r}$
$=0$ $\vec{E}_{cavity} = E\hat{r} = -\sigma/\epsilon_0\hat{r} \Rightarrow$ $\sigma$ uniform

ii) Suppose that another negatively charged particle is held outside the conductor in case B. Does the presence of this external change the charge density on the inner surface of the cavity for case B? Explain your answer.

No The previous argument applies regardless of what is outside the conductor. The potential in the cavity is only determined by the charge inside the cavity and the cavity radius. A point charge outside does not affect these $\Rightarrow$ does not affect cavity surface charge.

iii) Suppose that another negatively charged particle is held outside the conductor in case B and is moved closer to the conductor. Does the charge density on the inner surface of the cavity for case B depend on the location of the external charge? Explain your answer.

No. The same reason as ii

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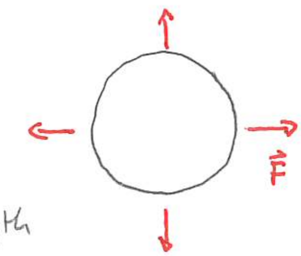
## Question 3

A circular ring with radius $R$ carries fixed charge that is uniformly distributed. The ring lies in the $xy$-plane with its center at the origin. It rotates with constant angular velocity $\omega$ about the $z$-axis.

a) Suppose that the ring is placed in a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ where $B$ is a constant. Is the force exerted by the magnetic field on the ring zero or not? Explain your answer.

$$\vec{F} = \int \vec{I} \times \vec{B} \, dl'$$




Here $\vec{I} = \lambda \vec{v} = \lambda \omega R \hat{\phi}$

where $\lambda$ is the charge per unit length

$$\vec{I} \times \vec{B} = \lambda \omega R B \hat{\phi} \times \hat{z} = \lambda \omega R B \hat{s}$$

These point radially out with uniform magnitude. Integration

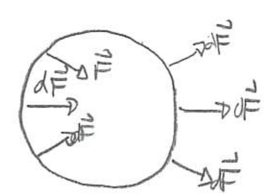
$\longrightarrow \vec{F} = 0$

b) Suppose that the ring is placed in a magnetic field $\mathbf{B} = B\cos\phi\hat{\mathbf{z}}$ where $B$ is a constant (these are cylindrical coordinates). Is the force exerted by the magnetic field on the ring zero or not? Explain your answer.

$$d\vec{F} = \vec{I} \times \vec{B} = \lambda \omega R B \cos\phi \, \hat{s}$$

These are illustrated

They do not cancel and add to $\longrightarrow$

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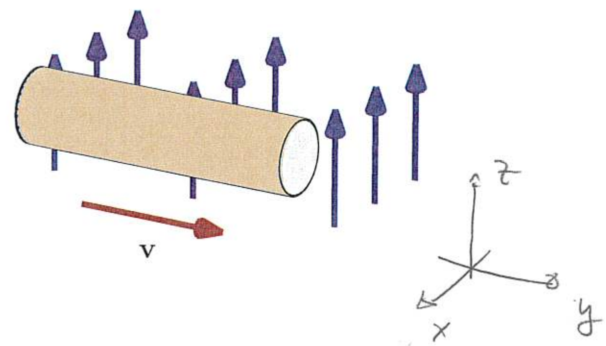
## Question 4

A cylindrical wire of radius $R$ and length $L$ has charge, $Q$, distributed uniformly over its surface. The charge is fixed relative to the surface of the wire. The cylinder moves with velocity $\mathbf{v}$ along the direction of its axis.




a) Determine an expression for the surface current density $\mathbf{K}$ for any point on the wire.

$\vec{K} = \sigma \vec{v}$

Then $\sigma = \dfrac{Q}{\text{curved area}} = \dfrac{Q}{2\pi R L}$

$\Rightarrow \vec{K} = \dfrac{Q}{2\pi R L} \vec{v}$

b) The wire moves through a region in which there is a uniform magnetic field, $\mathbf{B}$, that points in a direction perpendicular to the axis of the cylinder. Determine the net force exerted by the magnetic field on the wire.

$\vec{F} = \int \vec{K} \times \vec{B} \, da'$

Then using the illustrated axes $\quad \vec{K} \times \vec{B} = \dfrac{Q}{2\pi R L} v \hat{y} \times B\hat{z}$

$= \dfrac{QvB}{2\pi R L} \hat{x}$

So $\vec{F} = \displaystyle\int \frac{QvB}{2\pi R L} \hat{x} \, da'$

$= \dfrac{QvB}{2\pi R L} \hat{x} \displaystyle\int_{\text{curved surface}} da' = \dfrac{QvB}{2\pi R L} \hat{x} \, 2\pi R L = QvB \hat{x}$

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## Question 5

A infinitely long beam of charged particles moves with constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$. The beam extends out to infinity with a current density that approaches zero infinitely far from the $z$ axis. The current density in the beam is, in cylindrical coordinates,

$$\mathbf{J}(\mathbf{r}') = J_0 e^{-s'/a}\,\hat{\mathbf{z}}$$

where $a >$ is a constant with units of m and $J_0 > 0$ has units of $\mathrm{A/m^2}$.

3 a) Determine the total current in the beam (flowing in the $+\hat{\mathbf{z}}$ direction). Note that it is not infinite.

Integrate over an infinite plane perpendicular to $\hat{z}$

$$I = \int \vec{J}(\vec{r}') \cdot d\vec{a}'$$

$0 \le s' \le \infty$

$0 \le \phi' \le 2\pi$

$$\vec{J}(\vec{r}') \cdot d\vec{a}' = J_0 e^{-s'/a} s'\,ds'\,d\phi'$$

$d\vec{a}' = s'ds'd\phi'\hat{z}$

$$I = \int_0^{2\pi} d\phi' \int_0^{\infty} ds'\, s'\, J_0 e^{-s'/a} = 2\pi J_0 \int_0^{\infty} s' e^{-s'/a} ds'$$

$$I = 2\pi J_0 a^2 \Rightarrow J_0 = \frac{I}{2\pi a^2}$$

9 b) Determine the magnetic field produced by the beam at all points. Express the result in terms of the total current.

Ampère's Law

The magnetic has for $\vec{B} = B_s\hat{s} + B_\phi\hat{\phi} + B_z\hat{z}$.

By the Biot-Savart Law, there cannot be a component along $\vec{J}$ so cannot be along $\hat{z} \Rightarrow B_z = 0$

Rotating about the $\hat{x}$ axis through $180^\circ$ will transform $B_s \to B_s$, $B_\phi \to -B_\phi$

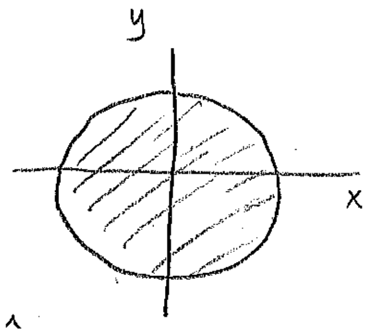
It will also transform $\vec{J} \to -\vec{J}$ and thus $\vec{B} \to -\vec{B}$.

This means $B_s = 0$

Thus $\vec{B} = B_\phi(s)\hat{\phi}$

Question 5 continued ...

Use an Amperian loop centered on $z$ in x-y plane with radius $s$. Then




$$\oint_{\text{loop}} \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int_{\text{loop surface}} \vec{J}\cdot d\vec{a}'$$

Now along the loop $0 \le \phi \le 2\pi$ $\quad d\vec{l} = s\, d\phi\, \hat{\phi}$

$$\Rightarrow \vec{B}\cdot d\vec{l} = B_\phi(s)\, s\, d\phi$$

$$\Rightarrow \int_0^{2\pi} B_\phi(s)\, s\, d\phi = \mu_0 \int \vec{J}\cdot d\vec{a}'$$

$$\Rightarrow B_\phi(s)\, s\, 2\pi = \mu_0 \int \vec{J}\cdot d\vec{a}' \Rightarrow B_\phi(s) = \frac{\mu_0}{2\pi s}\int \vec{J}\cdot d\vec{a}'$$

For the surface integral $\quad 0 \le s' \le s$, $0 \le \phi' \le 2\pi$ $\quad d\vec{a} = s'\, ds'\, d\phi'\, \hat{z}$

$$\Rightarrow \int \vec{J}\cdot d\vec{a} = \int_0^s ds' \int_0^{2\pi} d\phi'\, s'\, e^{-s'/a} J_0$$

$$= 2\pi J_0 \int_0^s s' e^{-s'/a}\, ds = 2\pi J_0 (-a)(s'+a) e^{-s'/a}\Big|_0^s$$

$$= 2\pi J_0 a \left[a - (s+a) e^{-s/a}\right]$$

Thus

$$B_\phi(s) = \frac{\mu_0}{2\pi s} 2\pi J_0 a \left[a - (s+a) e^{-s/a}\right] = \frac{\mu_0 I}{2\pi s\, a}\left[a(1-e^{-s/a}) - s e^{-s/a}\right]$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\left[(1-e^{-s/a}) - \frac{s}{a} e^{-s/a}\right]\hat{\phi}$$

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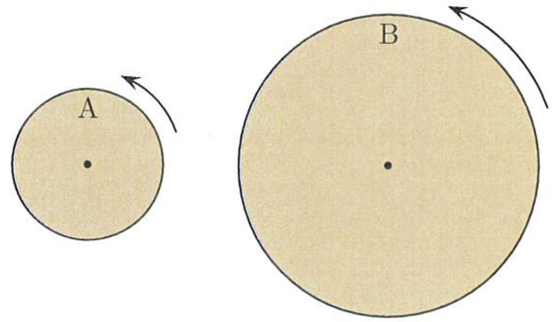
## Question 6




Two circular disks each carry uniformly distributed charge ~~total~~ and rotate about an axis through the center of the disk and perpendicular to the disk. They rotate with the same angular velocity and the total charge on each disk is the same. The radius of B is twice that of A. How does the magnetic dipole moment of B compare (larger, smaller, . . . ) to A? Explain your answer.

$$\vec{m} = \frac{1}{2}\int \vec{r}' \times \vec{K}\, da'$$

In these cases $\vec{K} = \sigma \vec{v}$ where $\sigma$ is the surface charge density

Then $\vec{v} = \vec{\omega} \times \vec{r} = \omega r \hat{\phi}$

$$\Rightarrow \vec{K} = \sigma \omega r \hat{\phi}$$

So

$$\vec{m} = \frac{1}{2}\int s'\hat{s} \times \sigma\omega s' \hat{\phi}\, da' = \frac{1}{2}\int \sigma\omega s'^2\, \hat{s} \times \hat{\phi}\, da'$$

$$\Rightarrow \vec{M} = \frac{\sigma\omega}{2}\int \underbrace{s'^2 da'}_{\to\, s'ds'd\phi'}\, \hat{z} = \frac{\sigma\omega}{2}\int_0^R ds' \int_0^{2\pi} d\phi'\, s'\, s'^2 \hat{z} = \frac{\cancel{2}\pi\sigma\omega}{\cancel{2}}\frac{R^3}{3}\hat{z}$$

$$\Rightarrow \vec{M} = \frac{\pi\sigma\omega}{3}R^3\hat{z} \qquad \text{But } \sigma = \frac{Q}{\pi R^2} \qquad \Rightarrow \vec{M} = \frac{Q\omega R}{3}\hat{z}$$

This is larger for B.

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## Question 7

A current produces magnetic vector potential, in Cartesian coordinates,

$$\mathbf{A} = \cancel{B}\beta \left(y^2\hat{\mathbf{x}} + x^2\hat{\mathbf{y}}\right)$$

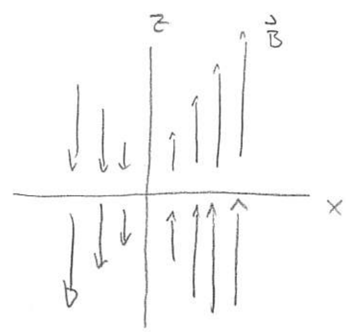
where $\cancel{B}\beta > 0$ is a constant.

a) Determine the magnetic field that is associated with $\mathbf{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ By^2 & Bx^2 & 0 \end{vmatrix} = \hat{z}\left(2x\beta - 2y\beta\right)$$

$$\Rightarrow \vec{B} = 2\beta(x-y)\hat{z}$$

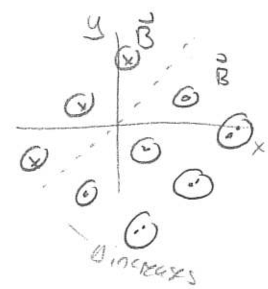


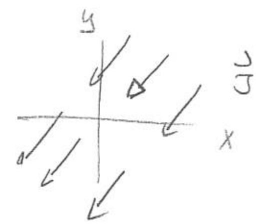

b) Determine the current density that produces this magnetic field.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \qquad \Rightarrow \vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right)$$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 2\beta(x-y) \end{vmatrix}$$

$$= \frac{1}{\mu_0}\left[-2\beta\hat{x} - 2\beta\hat{y}\right] = -\frac{2\beta}{\mu_0}\left(\hat{x} + \hat{y}\right)$$




$$\vec{J} = -\frac{2\beta}{\mu_0}\left(\hat{x} + \hat{y}\right)$$

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