# Electromagnetic Theory I: Class Exam I

2 October 2025

**Name:** ______________________________ **Total:** /50

## Instructions

- There are 6 questions on 8 pages.
- Show your reasoning and calculations and always explain your answers.

## Physical constants and useful formulae

| | |
|---|---|
| Permittivity of free space | $\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$ |
| Charge of an electron | $e = -1.60 \times 10^{-19}\,\mathrm{C}$ |

**Integrals**

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

## Question 1

A solid sphere with radius $R$ carries charge with density, in spherical coordinates,

$$\rho(\mathbf{r}') = \frac{\beta}{r'^2}$$

where $\beta > 0$ is a constant with units of C/m.

+3 a) Determine an expression for the total charge in the sphere, $Q$, in terms of $\beta, R$ and constants.

$$Q = \int \rho(\vec{r}')\, d\tau'$$

Here $$\left.\begin{array}{l} 0 \le r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} d\tau' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$Q = \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi' \cancel{r'^2} \sin\theta' \, \beta/\cancel{r'^2}$$

$$Q = R\, 4\pi \beta$$

Thus $$\beta = \frac{Q}{4\pi R}$$

b) Determine the electric field at all points inside and outside the sphere.

Gauss' Law states $$\oint \vec{E}\cdot d\vec{a} = q_{enc}/\epsilon_0$$

+1 ① Symmetry: rotating about any axis will not change the charge distribution but will change $E_\phi$ and $E_\theta$. Thus

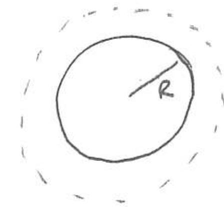
$$\vec{E} = E_r(r)\,\hat{r}$$

Question 1 continued ...

+5

(2) Gaussian surface = sphere centered at origin, radius $r$.

Then on the sphere




$$d\vec{a} = r^2 \sin\theta' \, d\theta' d\phi' \, \hat{r}$$

and $0 \le \theta' \le \pi$

$0 \le \phi' \le 2\pi$

So $\quad \vec{E} \cdot d\vec{a} = E_r(r) \, r^2 \sin\theta' \, d\theta' d\phi'$

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, E_r(r) r^2 \sin\theta' = E_r(r) r^2 \int_0^{\pi} \sin\theta' d\theta' \int_0^{2\pi} d\phi' = 4\pi E_r(r) r^2$$

Thus

$$4\pi r^2 E_r(r) = q_{enc}/\epsilon_0 \Rightarrow E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

+3

(3) Inside sphere $(r < R)$

$$q_{enc} = \int_0^r dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, \cancel{r'^2} \sin\theta' \, \beta / \cancel{r'^2} = r \, 4\pi\beta = \frac{r}{R} Q.$$

So

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{rQ}{R} \frac{1}{r^2} \hat{r} \Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{rR} \hat{r} \qquad r < R}$$

$$\text{OR} \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{4\pi\beta R}{rR} \hat{r} = \frac{\beta}{\epsilon_0 r} \hat{r}$$

+2

(4) Outside sphere $\quad r > R$

$$q_{enc} = Q \Rightarrow \boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} \qquad r > R}$$

$$\text{or} \quad \vec{E} = \frac{\beta R}{\epsilon_0 r^2} \hat{r}$$

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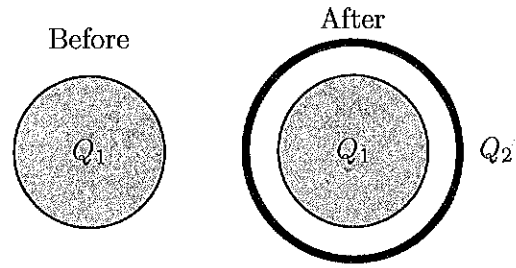
## Question 2

A solid sphere is uniformly charged with a total charge of $Q_1$. This produces an electric field everywhere. Subsequently a hollow spherical shell with uniformly distributed charge $Q_2$ is situated beyond the solid sphere without changing the distribution within the solid sphere.




a) Does the additional charge $Q_2$ alter the field *inside the solid sphere?* Explain your answer.

According to Gauss' Law $\oint \vec{E} \cdot \vec{dl} = \frac{q_{enc}}{\epsilon_0}$

If we used this with spherical symmetry the $\vec{E}$ only depends on the enclosed charge. If the surface is inside the solid the enclosed charge does not change. So <u>No</u> it does not change.

b) Does the additional charge $Q_2$ alter the field *between the spheres?* Explain your answer.

No, for the same reason as a)

c) Does the additional charge $Q_2$ alter the field *outside the hollow sphere?* Explain your answer.

<u>Yes</u>, Now the Gaussian surface includes the hollow sphere charge and $q_{enc}$ changes

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## Question 3

An infinitely long solid cylinder with radius $R$ carries charge whose density is given, in cylindrical coordinates, by

$$\rho(s', \phi', z') = \alpha s'$$

and $\alpha$ is a constant with units of C/m$^4$. Determine the electric field at all points *inside and outside* the cylinder.

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(1) Symmetry $\vec{E} = E_s \hat{s} + E_\phi \hat{\phi} + E_z \hat{z}$.

By rotating about any axis perpendicular to $z$ we change $E_\phi$ and $E_z$ but the charge density does not change. Thus $E_\phi = 0 = E_z$. So

$$\vec{E} = E_s(s)\, \hat{s}$$

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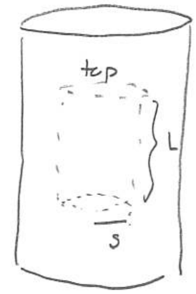
(2) Gaussian surface - cylinder with radius $s$ from $0 \le z \le L$




Then $\oint \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$

$$\Rightarrow \int_{top} \vec{E} \cdot d\vec{a} + \int_{side} \vec{E} \cdot d\vec{a} + \int_{bottom} \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$$

need this for 1pt

On the top and bottom $d\vec{a} = \pm da\, \hat{z}$ $\Rightarrow \vec{E} \cdot d\vec{a} = 0$. Thus

$$\int_{side} \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$$

On the side
$$\left.\begin{array}{l} s' = s \\ 0 \le \phi' \le 2\pi \\ 0 \le z' \le L \end{array}\right\} d\vec{a} = s'\, d\phi'\, dz'\, \hat{s} = s\, d\phi'\, dz'\, \hat{s}$$

Thus $\vec{E} \cdot d\vec{a} = E_s(s)\, s\, d\phi'\, dz'$

Question 3 continued ...

Thus

$$\int_{\text{sides}} \vec{E}\cdot d\vec{a} = \int_0^L\int_0^{2\pi} E_s(s)s\, d\phi' dz' = E_s(s)s\, 2\pi L.$$

so $E_s(s)s\, 2\pi L = q_{enc}/\epsilon_0 \Rightarrow \boxed{E_s(s) = \dfrac{q_{enc}}{2\pi\epsilon_0 s L}}$

(3) Enclosed charge $s < R$

Here

$$q_{enc} = \int \rho\, d\tau' = \int_0^s ds' \int_0^{2\pi} d\phi' \int_0^L dz'\, s' \underbrace{\rho(\vec{r}')}_{\beta s'}$$

$$= \alpha \cancel{\beta} \int_0^s s'^2 ds' \int_0^{2\pi} d\phi' \int_0^L dz' = \alpha 2\pi L\, s^3/3$$

+3

Thus $E_s(s) = \dfrac{\overset{\alpha}{\cancel{\beta}}\, \cancel{2\pi} \cancel{L}\, s^3/3}{\cancel{2\pi}\epsilon_0 s \cancel{L}} = \dfrac{\alpha\beta s^2}{3\epsilon_0} \Rightarrow \boxed{\vec{E} = \dfrac{\alpha s^2}{3\epsilon_0}\hat{s} \quad s < R}$

Enclosed charge $s \geq R$

$q_{enc} = \alpha 2\pi L\, R^3/3$

+3

$\Rightarrow E_s(s) = \dfrac{\alpha\cancel{\beta}\, \cancel{2\pi} \cancel{L}\, R^3}{3\, \cancel{2\pi}\epsilon_0 s \cancel{L}} \Rightarrow \boxed{\vec{E} = \dfrac{\alpha\cancel{\beta} R^3}{3\epsilon_0 s}\hat{s} \quad s \geq R}$

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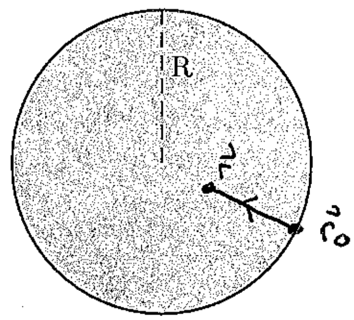
## Question 4

A solid sphere with radius $R$ contains charge $Q$; this is uniformly distributed. The field produced by the shell is (in spherical coordinates)

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon_0}\dfrac{r}{R^3}\,\hat{\mathbf{r}} & \text{if } r < R \\ \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r^2}\,\hat{\mathbf{r}} & \text{if } r > R \end{cases}$$




Determine the electrostatic potential at *all locations inside the sphere* assuming that $V = V_0$ at any point on the surface of the sphere.

Here

$$\Delta V = -\int_{\vec{r}}^{\vec{r}_0} \vec{E}\cdot d\vec{l} \quad \Rightarrow \quad V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

Take a radially inward path from the surface ($\vec{r}_0$) to an interior point ($\vec{r}$).
Then $V(\vec{r}_0) = V_0$. So

$$V(\vec{r}) = V_0 - \int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

Here $d\vec{l} = dr'\hat{r}$ $\qquad r \leq r' \leq R$, So

$$\vec{E}\cdot d\vec{l} = \frac{Q}{4\pi\epsilon_0}\frac{r'}{R^3}\,dr'$$

$$\Rightarrow \quad V(\vec{r}) = V_0 - \int_R^r \frac{Q}{4\pi\epsilon_0}\frac{r'}{R^3}\,dr' = V_0 - \frac{Q}{4\pi\epsilon_0}\frac{r'^2}{2R^3}\Big|_R^r$$

$$\Rightarrow \quad V(\vec{r}) = V_0 - \frac{Q}{R^3 8\pi\epsilon_0}\left[r^2 - R^2\right]$$

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## Question 5

Someone proposes to create a "quadratic electric" field using stationary charges so that

$$\mathbf{E} = \alpha\left(x^2\hat{\mathbf{x}} + y^2\hat{\mathbf{y}}\right)$$

where $\alpha > 0$ is a constant.

a) Is this a possible electrostatic field? Explain your answer.

$$\vec{\nabla}\times\vec{E} = \alpha\left[\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x^2\right]\hat{z} = 0 \qquad \text{Since } \vec{\nabla}\times\vec{E} = 0 \quad \underline{\text{Yes}}$$

b) What charge distribution might give rise to this electric field?

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \qquad \rho = \epsilon_0\vec{\nabla}\cdot\vec{E}$$
$$= \epsilon_0\left[\frac{\partial}{\partial x}(\alpha x^2) + \frac{\partial}{\partial y}(\alpha y^2)\right] = \alpha\epsilon_0(x+y)$$

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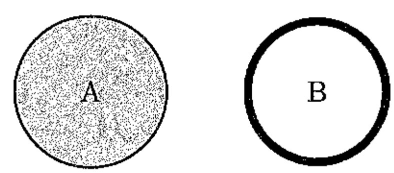
## Question 6

Consider two spheres, each with the same radius. The same charge is placed on each sphere. However, in sphere A, it is uniformly distributed throughout the sphere, while in sphere B it is uniformly distributed on the surface of the sphere. Consider the work done to assemble each charge distribution. Is the work done to assemble A the same as, larger than or smaller than that for B? Explain your answer. *Note: the diagram illustrates these near to each other; they are supposed to be considered separately.*




$$W = \underbrace{\frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau}_{\text{all space}} = \underbrace{\frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau}_{\text{inside sphere}} + \underbrace{\frac{\epsilon_0}{2}\int \vec{E}\cdot\vec{E}\,d\tau}_{\text{outside}}$$

Outside the sphere $\vec{E}$ is same. So that term is the same. For the hollow sphere $\vec{E} = 0$ inside. So that term gives 0. But it does not for the solid sphere.

So

$$W_{solid} > W_{hollow}$$

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<u>larger</u>