Electromagnetic Theory I: Final Exam

9 December 2025

Name: _____ **Total:** **/70**

Instructions

- There are 8 questions on 11 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C}^2/\mathrm{Nm}^2$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A}^2$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

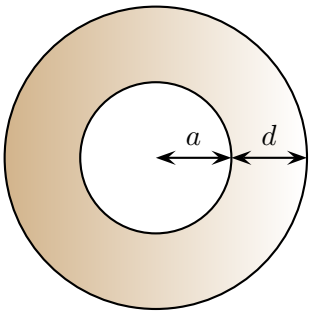
$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A hollow spherical shell with non-zero thickness has the illustrated dimensions and contains charge that is distributed with density (in spherical coordinates)



$$\rho(\mathbf{r}') = \begin{cases} \dfrac{\alpha}{4\pi r'^2 d} & \text{if } a < r < d \text{ and} \\ 0 & \text{if } r < a, \end{cases}$$

where α has units of C.

a) Determine an expression for the total charge, Q, contained in the shell.

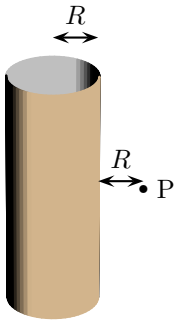
b) Determine an expression for the electric field produced by the charge distribution *at all points*. Express your results in terms of the total charge Q contained within the shell.

Question 1 continued ...

/15

Question 2

An infinitely long cylinder with radius R contains charge that is symmetrically distributed about the axis of the cylinder and varies with radial distance from the axis. There is no charge beyond the cylinder. The diagram illustrates a segment of the cylinder. Suppose that you wanted to used Gauss' Law to determine the field at the illustrated point P.



a) Describe the Gaussian surface that you would use to determine the field at P. Set up, but do not evaluate, the double integral required to evaluate $\oint \mathbf{E} \cdot \mathbf{da}$. Ensure that you include the integration variables and their limits (note that you do not have to evaluate the integral).

b) The enclosed charge that appears in Gauss's Law will have to be determined by integration. Describe the details of the integral, including its limits, that will give the enclosed charge (note that you do not have to evaluate the integral).

/4

Question 3

Answer **either part a) or part b) for full credit.** If you answer both, each will be graded and you will be awarded the highest score that you obtained for one of the parts.

a) The electrostatic potential produced by a particular stationary charge distribution is (in spherical coordinates)

$$V = V_0 e^{-r^2/\sigma}$$

where $\sigma > 0$ is constant with dimensions of m^2. Determine **the electric field** produced by this potential and **the charge distribution** that gives rise to this.

Question 3 continued ...

b) A hollow spherical shell with radius R holds charge Q that is uniformly distributed. The field produced by this is

$$\mathbf{E} = \begin{cases} 0 & \text{inside the sphere and} \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} & \text{outside the sphere.} \end{cases}$$

Determine the total work required to assemble this charge distribution.

/8

Question 4

A circular disk in the xy plane carries uniform surface charge density that is fixed to the surface of the disk. The disk rotates with constant angular velocity about the z-axis. This is then placed in an external magnetic field $\mathbf{B}$.

a) Suppose that $\mathbf{B} = B_0\hat{\mathbf{z}}$ where $B_0 > 0$ in constant. Which of the following (choose one) is true about the net force on the disk, $\mathbf{F}$?

i) $\mathbf{F} = 0$.
ii) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{s}}$ direction.
iii) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{x}}$ direction.
iv) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{y}}$ direction.

Explain your choice.

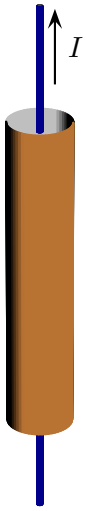
b) Suppose that (in cylindrical coordinates) $\mathbf{B} = B_0 \cos\phi\,\hat{\mathbf{z}}$ where $B_0 > 0$ in constant. Which of the following (choose one) is true about the net force on the disk, $\mathbf{F}$?

i) $\mathbf{F} = 0$.
ii) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{s}}$ direction.
iii) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{x}}$ direction.
iv) $\mathbf{F} \neq 0$ and is along the $\hat{\mathbf{y}}$ direction.

Explain your choice.

/6

Question 5



An infinitely long wire lies along the z axis. An infinitely long cylindrical shell with radius R has its axis along the z axis. The wire carries current I. Current with the same magnitude flows in the opposite direction down the cylindrical shell and is uniformly distributed along the shell.

a) Determine the surface current density along the cylindrical shell.

b) Determine the magnetic field produced by the entire arrangement of currents at all locations.

Question 5 continued ...

/15

Question 6

An infinitely long solenoid carries current I and has n turns per unit length. Which of the following is a possible vector potential (in cylindrical coordinates) for the magnetic field *inside* the solenoid?

i) $\mathbf{A} = \alpha s\,\hat{\boldsymbol{\phi}}$ where α is constant.
ii) $\mathbf{A} = \beta\,\hat{\boldsymbol{\phi}}$ where β is constant.
iii) $\mathbf{A} = \gamma\,\hat{\mathbf{z}}$ where γ is constant.
iv) $\mathbf{A} = \tau s\,\hat{\mathbf{z}}$ where τ is constant.

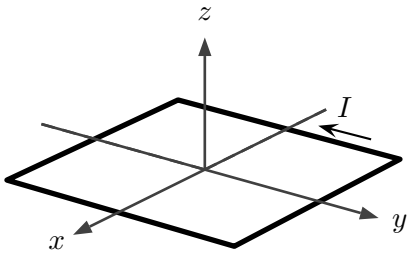
Explain your choice.

/6

Question 7

Answer **either part a) or part b) for full credit.** If you answer both, each will be graded and you will be awarded the highest score that you obtained for one of the parts.

a) A square loop whose sides have length a lies in the xy plane. Determine the magnetic dipole moment of this loop, the magnetic vector potential of the loop and use the result to determine the magnetic field produced by the loop (in spherical coordinates).



Question 7 continued ...

b) A solid sphere with radius R is uniformly charged with total charge Q. The sphere rotates about the z axis with angular velocity ω. Determine expressions for the charge and current density and show that they satisfy the continuity equation.

/8

Question 8

A very long solenoid with radius R has n turns per unit length and is oriented with its axis along the z axis. The magnetic field is

$$\mathbf{B} = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & \text{inside and} \\ 0 & \text{outside,} \end{cases}$$

and the current varies with time according to $I = I_0 \cos(\omega t)$ where $\omega > 0$ is a constant with units of s^{-1}. The solenoid is surrounded by a single loop with radius $a > R$ and whose axis is also along the z axis. Determine the EMF around the loop.

/8