Phys 311
Fall 2020

Electromagnetic Theory: Final Exam

7 December 2020

Name: Solution **Total:** **/60**

Instructions

- There are 8 questions on 12 pages.
- Show your reasoning and calculations and always explain your answers.

Important: This is an "open book" exam. You are not allowed to consult any person about the content of the exam except the course instructor. You cannot use any existing (prior to the moment that you provide your response to that problem) solution to a problem whose wording is the same as that question on this exam.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

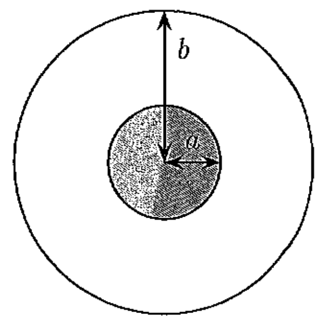
$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A solid sphere with radius a is surrounded by a concentric hollow spherical with radius b. The solid sphere holds stationary charge with density, in spherical coordinates, in spherical coordinates

$$\rho(r', \theta', \phi') = \alpha r'$$

where $\alpha > 0$ is a constant. The hollow spherical shell holds uniformly charge Q_b and this is uniformly distributed.



a) Determine the total charge Q_a on the inner solid sphere.

$$Q_a = \int \rho \, d\tau$$

$$= \int_0^a dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' r'^2 \sin\theta' \, \alpha r'$$

$$\left.\begin{array}{l} 0 \le r' \le a \\ 0 \le \phi' \le 2\pi \\ 0 \le \theta' \le \pi \end{array}\right\} \quad d\tau = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$$

$$= \alpha \underbrace{\int_0^a r'^3 dr'}_{\frac{a^4}{4}} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta' d\theta'}_{2} = \alpha a^4 \pi$$

$$\Rightarrow \quad Q_a = \alpha a^4 \pi$$

b) Determine the electric field at all points: inside the solid sphere, between the two sphere and beyond the spherical shell. Express the results in terms of Q_a and Q_b.

Gauss' law $\quad \int \vec{E} \cdot d\vec{a} = q_{enc}/\epsilon_0$

$$\vec{E} = E_r \hat{r} + E_\phi \hat{\phi} + E_\theta \hat{\theta}$$

By symmetry (rotate about axis in xy plane) $\quad E_\phi = 0$, $E_\theta = 0$

Then E_r can only depend on r

$$\Rightarrow \vec{E} = E_r(r) \hat{r}$$

Question 1 continued ...

Gaussian surface is sphere of radius r

Then
$$\left.\begin{array}{l} r' = r \\ 0 \leq \phi' \leq 2\pi \\ 0 \leq \theta' \leq \pi \end{array}\right\} \quad \begin{aligned} d\vec{a} &= r'^2 \sin\theta' \, d\phi' d\theta' \, \hat{r} \\ &= r^2 \sin\theta' \, d\phi' d\theta' \, \hat{r} \end{aligned}$$

So $\vec{E} \cdot d\vec{a} = r^2 E_r(r) \sin\theta' \, d\phi' \, d\theta'$

$$\oint \vec{E} \cdot d\vec{a} = \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \, r^2 E_r = 4\pi r^2 E_r$$

$$\Rightarrow \quad E_r = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

Case 1 $\quad 0 \leq r \leq a \quad$ Here $\quad q_{enc} = \int_0^r dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \, r'^2 \sin\theta' \, r' \alpha$

$$= \alpha r^4 \pi = Q_a \frac{r^4}{a^4}$$

So $\quad E_r = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q_a r^2}{a^4}$

Case 2 $\quad a < r \leq b \quad q_{enc} = Q_a \quad E_r = \frac{1}{4\pi\epsilon_0} \frac{Q_a}{r^2}$

Case 3 $\quad b \leq r \quad q_{enc} = Q_a + Q_b \quad E_r = \frac{1}{4\pi\epsilon_0} \frac{Q_a + Q_b}{r^2}$

Thus
$$\vec{E} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{Q_a r^2}{a^4} \hat{r} & r \leq a \\ \frac{1}{4\pi\epsilon_0} \frac{Q_a}{r^2} \hat{r} & a < r \leq b \\ \frac{1}{4\pi\epsilon_0} \frac{Q_a + Q_b}{r^2} \hat{r} & b \leq r \end{cases}$$

/12

Question 2

Gauss' Law is often expressed as

$$\oint \mathbf{E} \cdot \mathrm{da} = \frac{Q_{\text{enc}}}{\epsilon_0}.$$

Which of the following (choose one) can be used to obtain this version of Gauss' Law?

i) $\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ and the divergence theorem.

ii) $\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ and Stokes' theorem.

iii) $\boldsymbol{\nabla} \times \mathbf{E} = 0$ and the divergence theorem.

iv) $\boldsymbol{\nabla} \times \mathbf{E} = 0$ and Stokes' theorem.

$$\underbrace{\int \vec{\nabla} \cdot \vec{E}\, d\tau}_{\text{closed region}} = \frac{1}{\epsilon_0} \underbrace{\int \rho\, d\tau}_{q_{enc}}$$

$$\underbrace{\oint \vec{E} \cdot d\vec{a}}_{\text{surface}} = \frac{q_{enc}}{\epsilon_0}$$

/2

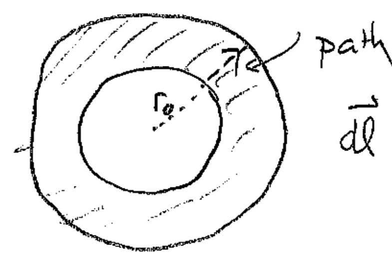
Question 3

An infinitely long cylindrical pipe has walls with an inner radius r_0 and an outer radius of r_1. The pipe carries a charge whose density only depends on radial distance. It emerges that the electric field in the region within the walls of the pipe is, in cylindrical coordinates,

$$\mathbf{E} = \rho \frac{s^2 - r_0^2}{2\epsilon_0 s} \hat{\mathbf{s}}$$

where $\rho > 0$ is a constant. Determine the electric potential difference between one wall of the cylinder and the other.

$$\Delta V = - \int \vec{E} \cdot d\vec{l}$$



$$d\vec{l} = ds\, \hat{s}$$

$$\vec{E} \cdot d\vec{l} = \rho \frac{s^2 - r_0^2}{2\epsilon_0 s} ds$$

$$= \frac{\rho}{2\epsilon_0} \left[s - \frac{r_0^2}{s} \right] ds$$

$$\Delta V = - \int_{r_0}^{r_1} \frac{\rho}{2\epsilon_0} \left[s - \frac{r_0^2}{s} \right] ds$$

$$\Delta V = -\frac{\rho}{2\epsilon_0} \left[\frac{s^2}{2} - r_0^2 \ln s \right]_{r_0}^{r_1}$$

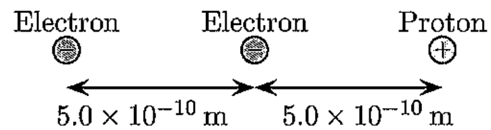
$$\Delta V = -\frac{\rho}{2\epsilon_0} \left[\frac{r_1^2 - r_0^2}{2} - r_0 \ln\left(\frac{r_1}{r_0}\right) \right]$$

/6

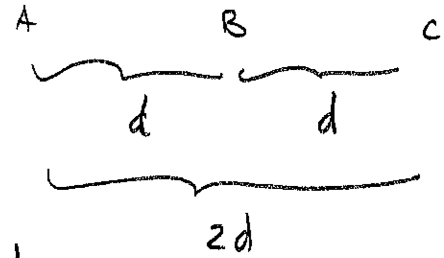
Question 4

In the following question, **either do part a) or part b) for full credit.** If you attempt both, you will be given credit equal to the highest score attained for either part.

a) Two electrons are each brought from infinitely far away to within the vicinity of a proton and held at rest in the indicated configuration. Determine the work to assemble this arrangement.



$$W = \sum_{i\neq j} \frac{q_i q_j}{4\pi\epsilon_0 r_{ij}}$$



$$= \frac{q_A q_B}{4\pi\epsilon_0}\frac{1}{d} + \frac{q_A q_C}{4\pi\epsilon_0}\frac{1}{2d} + \frac{q_B q_C}{4\pi\epsilon_0}\frac{1}{d}.$$

$$= \frac{1}{4\pi\epsilon_0 d}\left\{ \cancel{e^2} - \frac{e^2}{2} - \cancel{e^2} \right\}$$

$$= -\frac{e^2}{4\pi\epsilon_0\, 2d} = \frac{-(1.6\times 10^{-19}\,\cancel{C})^2}{4\pi \times 8.85\times 10^{-12}\,\cancel{C^2}/N\cancel{m^2} \times 2\times 5.0\times 10^{-10}\,\cancel{m}}$$

$$W = -2.3\times 10^{-19}\,J$$

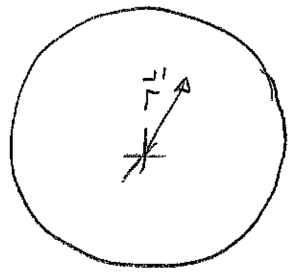
Question 4 continued ...

b) A solid sphere with radius R contains charge with density, in spherical coordinates

$$\rho(r', \theta', \phi') = \rho_0 \sin\phi'$$

where ρ_0 is a constant. Determine the electric dipole moment of this distribution.

$$\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\tau'$$



Here $\vec{r}\,' = r'\,\hat{r}$

$$\vec{p} = \int r'\rho_0 \sin\phi'\,\hat{r}\,d\tau'$$

$$\left.\begin{array}{l} 0 \le r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\}$$

$$d\tau' = r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$\vec{p} = \int_0^R dr'\,r'^3 \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'\;\rho_0 \sin\phi'\;\hat{r}$$

$$= \rho_0 \frac{R^4}{4} \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \sin\phi' \big[\sin\theta' \cos\phi'\,\hat{x} + \sin\theta' \sin\phi'\,\hat{y} + \cos\theta'\,\hat{z} \big]$$

$\leftarrow$ integrates to zero over ϕ'

$$= \rho_0 \frac{R^4}{4} \left[\cancelto{0}{\int_0^{2\pi} \sin\phi' \cos\phi'\,d\phi'} \int_0^{\pi} \sin^2\theta'\,d\theta'\,\hat{x} + \underbrace{\int_0^{2\pi} \sin^2\phi'\,d\phi'}_{\pi} \underbrace{\int_0^{\pi} \sin^2\theta'\,d\theta'}_{\pi/2}\,\hat{y} \right]$$

$$\vec{p} = \rho_0 \frac{R^4 \pi^2}{8}\,\hat{y}$$

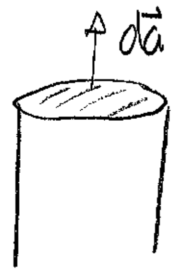
/7

Question 5

An infinitely long cylinder with radius R carries current whose density is, in cylindrical coordinates,

$$\mathbf{J}(s', \phi', z') = \frac{2I_0 s'^2}{\pi R^4}\,\hat{\mathbf{z}}.$$

a) Show that the total current along the cylinder is I_0.



$$I = \int \vec{J}\cdot d\vec{a}$$

$$= \frac{2I_0}{\pi R^4}\int_0^R s'^3 ds' \int_0^{2\pi} d\phi'$$

$$= \frac{\cancel{2} I_0}{\cancel{\pi R^4}}\,\frac{\cancel{R^4}}{\cancel{4}}\cdot\cancel{2\pi}$$

$$= I_0 \checkmark$$

$$d\vec{a} = s'ds'd\phi'\,\hat{z}$$

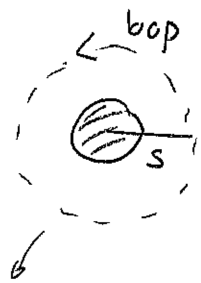
$$0 < s' \leq R$$

$$0 \leq \phi' \leq 2\pi$$

b) Determine the magnetic field that is produced by the current at any point outside the cylinder.

Ampère's law $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

$$\vec{B} = \underbrace{B_s\hat{s}}_{0 \text{ rotate } \pi \text{ about } \hat{x}} + B_\phi\hat{\phi} + \underbrace{\cancel{B_z\hat{z}}}_{0 \text{ by Biot-Sav.}} \Rightarrow \vec{B} = B_\phi(s)\,\hat{\phi}$$



$$\oint \vec{B}\cdot d\vec{l} = \int_0^{2\pi} B_\phi(s)\, s\, d\phi' = 2\pi s B_\phi(s) = \mu_0 I_{enc}$$

$$d\vec{l} = s\,d\phi'\,\hat{\phi}$$

But $I_{enc} = I_0$

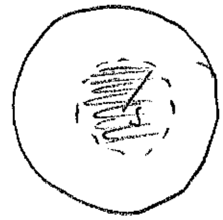
$$\Rightarrow B_\phi(s) = \frac{\mu_0 I_0}{2\pi s} \Rightarrow \vec{B} = \frac{\mu_0 I_0}{2\pi s}\,\hat{\phi}$$

Question 5 continued …

c) Determine the magnetic field that is produced by the current at any point inside the cylinder.

The argument for the l.h.s is the same. So

$$B_\phi(s) = \frac{\mu_0 I_{enc}}{2\pi s}$$



$$d\vec{a} = s'ds'd\phi'\hat{z}$$

Then $I_{enc} = \int \vec{J} \cdot d\vec{a}$ over shaded surface

$$= \int_0^s s'ds' \int_0^{2\pi} d\phi' \frac{2 I_0 s'^2}{\pi R^4}$$

$$= \frac{2 I_0}{\pi R^4} \int_0^s s'^3 ds' \int_0^{2\pi} d\phi'$$

$$= \frac{\cancel{2} I_0}{\cancel{\pi} R^4} \frac{s^4}{\cancel{4}} \cancel{2\pi} = I_0 \frac{s^4}{R^4}$$

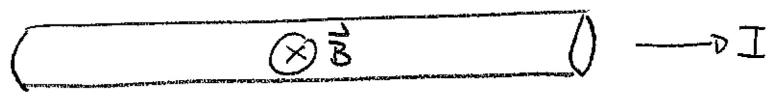
$$\Rightarrow B_\phi(s) = \frac{\mu_0 s^3 I}{2\pi R^4}$$

$$\vec{B} = \frac{\mu_0 s^3 I}{2\pi R^4} \hat{\phi}$$

/12

Question 6

Charged particles flow along the length of a horizontal cylindrical wire. The current is known to point right. Describe *how* you could use a transverse magnetic field to determine whether the charged particles are positive or negative. Your answer needs to contain a description of what you would measure, what the results might indicate and a description using physics that supports your scheme.



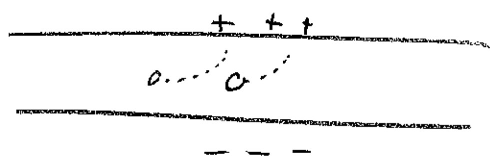
The field exerts a force $\vec{F} = q\vec{v} \times \vec{B}$ on any particle.

Suppose moving charges are positive. Then

$\vec{v}$ is $\rightarrow$

$\vec{v} \times \vec{B}$ is $\uparrow$

$\vec{F}$ is $\uparrow$ charges move



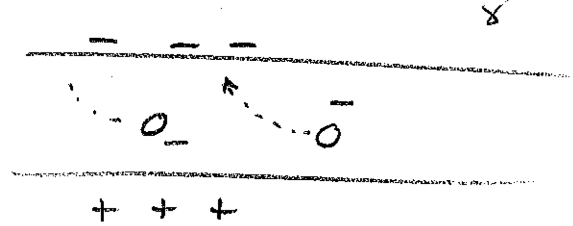
Suppose moving charges are negative. Then

$\vec{v}$ is $\leftarrow$

$\vec{v} \times \vec{B}$ is $\downarrow$

$\vec{F}$ is $\uparrow$

charges move



check which of these happen.

Measure potential diff across transverse section of wire

/6

Question 7

In the following question, **either do part a) or part b) for full credit.** If you attempt both, you will be given credit equal to the highest score attained for either part.

a) An infinitely long cylinder with radius R carries fixed charge. The cylinder is dragged along its axis (the z axis) with constant speed v and, in cylindrical coordinates, the charge density is

$$\rho = \rho_0 \frac{(z - vt)^2}{a^2}$$

where $\rho_0 > 0$ and $a > 0$ are constant. Determine an expression for the current density and show that this satisfies the continuity equation.

$$\vec{J} = \rho \vec{v}$$

$$= \rho_0 \frac{(z-vt)^2}{a^2} v \hat{z}$$

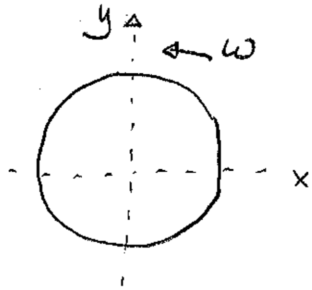
$$\vec{\nabla} \cdot \vec{J} = \frac{\partial J_z}{\partial z} = \frac{\partial}{\partial z}\left(\rho_0 \frac{(z-vt)^2}{a^2} v\right) = \frac{\rho_0 v}{a^2} 2(z-vt)$$

$$\frac{\partial \rho}{\partial t} = \frac{\rho_0}{a^2} 2(-v)(z-vt)$$

So $\quad \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

Question 7 continued . . .

b) A circular ring that lies in the xy plane has radius R and fixed charge with uniform density λ. The ring rotates about the z axis with speed v. Determine an expression for the magnetic dipole moment of the ring and use this to determine expressions for the magnetic vector potential and ~~approximate magnetic field~~ (both in terms of spherical coordinates) produced by the ring.



$$\vec{m} = \frac{1}{2}\int \vec{r}\,' \times \vec{I}\, dl'$$

Here $\quad \vec{I} = \vec{v}\lambda$

$$= R\omega\lambda\,\hat{\phi} \qquad \text{since } \vec{v} = R\omega\,\hat{\phi}$$

Then $\vec{r}\,' = r'\,\hat{r}$ gives

$$\vec{m} = \frac{1}{2}\int \underbrace{r'}_{\text{or } r'=R} R\omega\ \hat{r}\times\hat{\phi}\ dl$$

$$= \frac{1}{2}R^2\omega \int \underbrace{\hat{r}\times\hat{\phi}}_{\hat{z}}\ dl \qquad dl = R\,d\phi'$$

$$= \frac{1}{2}R^2\omega \int_0^{2\pi} R\,d\phi'\ \hat{z}$$

$$\vec{m} = \pi R^3\omega\ \hat{z}$$

$$\hat{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\pi R^3\omega\ \frac{\hat{z}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0 R^3\omega}{4r^2}(-\sin\theta)\,\hat{\phi}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{1}{r\sin\theta}\frac{\mu_0 R^3\omega}{4r^2}\overset{2\sin\theta\cos\theta}{\cancel{\frac{\partial}{\partial\theta}\sin^2\theta}}\ \hat{r}$$

$$+ \frac{\mu_0 R^3\omega}{4r^3}\sin\theta\ \overset{1}{\cancel{\frac{\partial}{\partial r}(r)}}\ \hat{\theta}$$

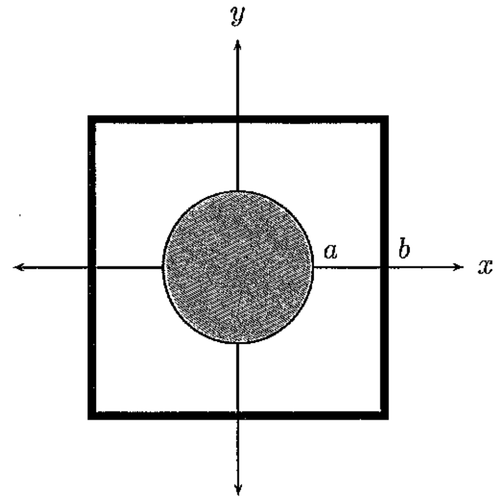
/7

Question 8

A magnetic field is produced in such a way that it is non-zero within a cylindrical region of radius a running along the z axis and is zero beyond this. Within the cylindrical region,

$$\mathbf{B} = B_0 e^{-t/\tau}\left(\frac{3}{5}\,\hat{\mathbf{x}} + \frac{4}{5}\,\hat{\mathbf{z}}\right)$$

where B_0 and τ are constants. A square loop of sides b lies stationary in the xy plane as illustrated.



a) Determine the EMF induced in the loop.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\Phi = \int \vec{B}\cdot d\vec{a}$$ only shaded area contributes

$d\vec{a} = s'ds'd\phi'\,\hat{z}$ $\quad 0 \leq s' \leq a$

$\quad 0 \leq \phi \leq 2\pi$

$$= \int_0^a ds' \int_0^{2\pi} d\phi' s' \; B_0 e^{-t/\tau}\,\frac{4}{5}$$

$$= B_0 e^{-t/\tau}\,\frac{4}{5}\int_0^a s'ds' \int_0^{2\pi} d\phi' = B_0 e^{-t/\tau}\,\frac{4\pi a^2}{5}$$

Then $\quad \frac{d\Phi}{dt} = -\frac{1}{\tau} B_0 e^{-t/\tau}\,\frac{4\pi a^2}{5} \Rightarrow \mathcal{E} = \frac{4 B_0 \pi a^2}{5\tau} e^{-t/\tau}$

b) Suppose that the square loop were located so that it was not centered at the origin but that the region of non-zero magnetic field was still located within the loop. Describe whether the induced EMF would be any different to that calculated above and, if so, how you would calculate it.

No, Flux is still same

/8