Electromagnetic Theory: Final Exam

7 December 2020

Name: ____________________________ **Total:** **/60**

Instructions

- There are 8 questions on 12 pages.
- Show your reasoning and calculations and always explain your answers.

Important: This is an "open book" exam. You are not allowed to consult any person about the content of the exam except the course instructor. You cannot use any existing (prior to the moment that you provide your response to that problem) solution to a problem whose wording is the same as that question on this exam.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

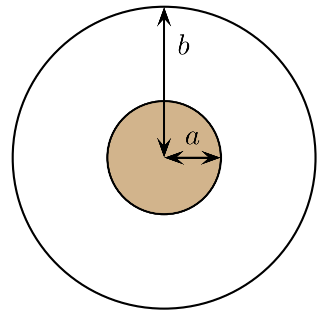
$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

A solid sphere with radius a is surrounded by a concentric hollow spherical with radius b. The solid sphere holds stationary charge with density, in spherical coordinates, in spherical coordinates

$$\rho(r', \theta', \phi') = \alpha r'$$

where $\alpha > 0$ is a constant. The hollow spherical shell holds uniformly charge Q_b and this is uniformly distributed.



a) Determine the total charge Q_a on the inner solid sphere.

b) Determine the electric field at all points: inside the solid sphere, between the two sphere and beyond the spherical shell. Express the results in terms of Q_a and Q_b.

Question 1 continued . . .

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Question 2

Gauss' Law is often expressed as

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{Q_{\mathrm{enc}}}{\epsilon_0}.$$

Which of the following (choose one) can be used to obtain this version of Gauss' Law?

i) $\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ and the divergence theorem.

ii) $\boldsymbol{\nabla} \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ and Stokes' theorem.

iii) $\boldsymbol{\nabla} \times \mathbf{E} = 0$ and the divergence theorem.

iv) $\boldsymbol{\nabla} \times \mathbf{E} = 0$ and Stokes' theorem.

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Question 3

An infinitely long cylindrical pipe has walls with an inner radius r_0 and an outer radius of r_1. The pipe carries a charge whose density only depends on radial distance. It emerges that the electric field in the region within the walls of the pipe is, in cylindrical coordinates,

$$\mathbf{E} = \rho \frac{s^2 - r_0^2}{2\epsilon_0 s} \hat{\mathbf{s}}$$

where $\rho > 0$ is a constant. Determine the electric potential difference between one wall of the cylinder and the other.

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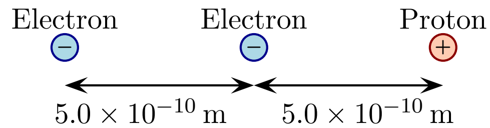
Question 4

In the following question, **either do part a) or part b) for full credit.** If you attempt both, you will be given credit equal to the highest score attained for either part.

a) Two electrons are each brought from infinitely far away to within the vicinity of a proton and held at rest in the indicated configuration. Determine the work to assemble this arrangement.



Question 4 continued . . .

b) A solid sphere with radius R contains charge with density, in spherical coordinates

$$\rho(r', \theta', \phi') = \rho_0 \sin \phi'$$

where ρ_0 is a constant. Determine the electric dipole moment of this distribution.

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Question 5

An infinitely long cylinder with radius R carries current whose density is, in cylindrical coordinates,

$$\mathbf{J}(s', \phi', z') = \frac{2I_0 s'^2}{\pi R^4}\,\hat{\mathbf{z}}.$$

a) Show that the total current along the cylinder is I_0.

b) Determine the magnetic field that is produced by the current at any point outside the cylinder.

Question 5 continued ...

c) Determine the magnetic field that is produced by the current at any point inside the cylinder.

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Question 6

Charged particles flow along the length of a horizontal cylindrical wire. The current is known to point right. Describe *how* you could use a transverse magnetic field to determine whether the charged particles are positive or negative. Your answer needs to contain a description of what you would measure, what the results might indicate and a description using physics that supports your scheme.

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Question 7

In the following question, **either do part a) or part b) for full credit.** If you attempt both, you will be given credit equal to the highest score attained for either part.

a) An infinitely long cylinder with radius R carries fixed charge. The cylinder is dragged along its axis (the z axis) with constant speed v and, in cylindrical coordinates, the charge density is

$$\rho = \rho_0 \frac{(z - vt)^2}{a^2}$$

where $\rho_0 > 0$ and $a > 0$ are constant. Determine an expression for the current density and show that this satisfies the continuity equation.

Question 7 continued ...

b) A circular ring that lies in the xy plane has radius R and fixed charge with uniform density λ. The ring rotates about the z axis with speed v. Determine an expression for the magnetic dipole moment of the ring and use this to determine expressions for the magnetic vector potential and approximate magnetic field (both in terms of spherical coordinates) produced by the ring.

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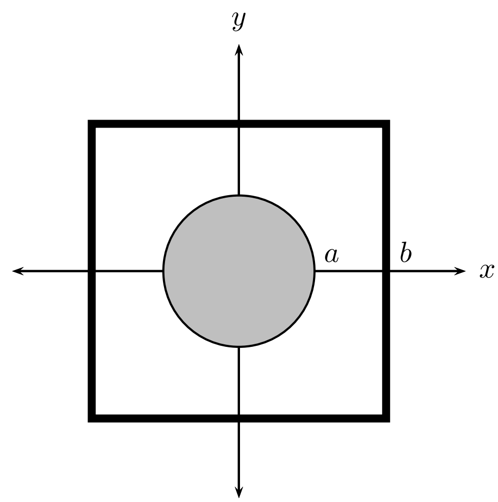
Question 8

A magnetic field is produced in such a way that it is non-zero within a cylindrical region of radius a running along the z axis and is zero beyond this. Within the cylindrical region,

$$\mathbf{B} = B_0 e^{-t/\tau} \left(\frac{3}{5}\,\hat{\mathbf{x}} + \frac{4}{5}\,\hat{\mathbf{z}} \right)$$

where B_0 and τ are constants. A square loop of sides b lies stationary in the xy plane as illustrated.



a) Determine the EMF induced in the loop.

b) Suppose that the square loop were located so that it was not centered at the origin but that the region of non-zero magnetic field was still located within the loop. Describe whether the induced EMF would be any different to that calculated above and, if so, how you would calculate it.

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