Phys 312 2025

Lecture 22

Fri: HW by 5pm

Tues: Read 11.1.1, 11.1.2

Potentials for a moving point charge

Suppose that a point charge with charge q moves along trajectory $\vec{w}(t)$.
We can determine the potentials at point $\vec{r}$ at time t as follows:

① Determine retarded time t_r by solving

$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

② Determine retarded position by

$$\vec{r}_r = \vec{w}(t_r)$$

and retarded separation vector

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$$

③ Calculate scalar potential

$$V(\vec{r},t) = \frac{q}{4\pi\epsilon_0} \frac{c}{\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v}}$$

where $\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r}$

④ Calculate vector potential

$$\vec{A}(\vec{r},t) = \frac{\vec{v}(t_r)}{c^2} V(\vec{r},t)$$

Liénard - Wiechert potentials

Fields produced by a moving point charge

The fields are given by a standard method.

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

This entails differentiation. For the scalar potential, the derivatives are with respect to x, y, z and t and the crucial terms will be

differentiate $\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v}$ and $\vec{v}(t_r)$

Now $\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$ and so when differentiating we need to

differentiate <u>explicitly</u> w.r.t to x, y, z
$\rightsquigarrow$ differentiate $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

differentiate <u>implicitly</u> with respect to x, y, z, t
$\rightsquigarrow$ differentiate t_r w.r.t x, y, z, t

Differentiate <u>implicitly</u> in $\vec{v}(t_r)$

Since the variables all appear in t_r via

$\vec{r}$ here

$$c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$$

The relevant derivatives appear as follows (Proofs Later)

Lemma 1

$$\vec{\nabla} t_r = \frac{-\vec{\imath}_r}{\imath_r c - \vec{v}\cdot\vec{\imath}_r}$$

Lemma 2

$$\vec{\nabla}(\vec{\imath}_r\cdot\vec{v}) = \vec{v} + (\vec{\imath}_r\cdot\vec{a} - v^2)\,\vec{\nabla} t_r$$

where the retarded acceleration is

$$\vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$$

Lemma 3

$$\frac{\partial t_r}{\partial t} = \frac{\imath_r c}{c\,\imath_r - \vec{v}\cdot\vec{\imath}_r}$$

Lemma 4

$$\frac{\partial}{\partial t}(\vec{v}\cdot\vec{\imath}_r) = \frac{\partial t_r}{\partial t}\left(\vec{\imath}_r\cdot\vec{a} - v^2\right)$$

<u>Lemma: 1:</u> $\vec{\nabla} t_r = \dfrac{-\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v}}$

<u>Proof:</u> The retarded time depends implicitly on $\vec{r}$. Specifically

$$c(t-t_r) = \mathscr{r}_r$$

$$\Rightarrow \quad c(t-t_r) = \left(\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r\right)^{1/2}$$

$$\Rightarrow \quad \vec{\nabla}\left(c(t-t_r)\right) = \vec{\nabla}\left(\ldots\right)^{1/2}$$

$$\Rightarrow \quad -c\vec{\nabla} t_r = \frac{1}{2}\frac{1}{\left[\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r\right]^{1/2}} \vec{\nabla}\left(\vec{\mathscr{r}}_r \cdot \vec{\mathscr{r}}_r\right)$$

$$\Rightarrow \quad -c\vec{\nabla} t_r = \frac{1}{2\mathscr{r}_r}\Big\{ \vec{\mathscr{r}}_r \times (\vec{\nabla}\times\vec{\mathscr{r}}_r) + \vec{\mathscr{r}}_r \times (\vec{\nabla}\times\vec{\mathscr{r}}_r) + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})(\vec{\mathscr{r}}_r)\Big\}$$

$$\Rightarrow \quad -c\vec{\nabla} t_r = \frac{1}{\mathscr{r}_r}\left[\vec{\mathscr{r}}_r \times (\vec{\nabla}\times\vec{\mathscr{r}}_r) + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r\right]$$

We will show:

(A): $\vec{\nabla} \times \vec{\mathscr{r}}_r = \vec{v} \times \vec{\nabla} t_r$

(B): $(\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r - \vec{v}\left[\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r)\right]$

First to prove (A):

$$\vec{\nabla} \times \vec{\mathscr{r}}_r = \vec{\nabla} \times \left(\vec{r} - \vec{w}(t_r)\right)$$

$$= \cancel{\vec{\nabla} \times \vec{r}} - \vec{\nabla} \times \vec{w}(t_r)$$

$$= -\vec{\nabla} \times \vec{w}(t_r)$$

But $\vec{\nabla} \times \vec{W} = \left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)\hat{z} + \left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right)\hat{x} + \left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}\right)\hat{y}$

and
$$\frac{\partial w_y}{\partial x} = \frac{\partial w_y}{\partial t_r}\frac{\partial t_r}{\partial x}$$
$$= \left.\frac{d w_y}{dt}\right|_{t_r}\frac{\partial t_r}{\partial x} \quad \text{etc,...}$$

gives:
$$\vec{\nabla} \times \vec{W} = \left(\frac{dw_y}{dt}\frac{\partial t_r}{\partial x} - \frac{dw_x}{dt}\frac{\partial t_r}{\partial y}\right)\hat{z} + \cdots$$
$$= \left[\left.\frac{dw_y}{dt}\right|_{t_r}(\vec{\nabla} t_r)_x - \left.\frac{dw_x}{dt}\right|_{t_r}(\vec{\nabla} t_r)_y\right]\hat{z} + \cdots$$
$$= \vec{\nabla}(t_r) \times \left.\frac{d\vec{w}}{dt}\right|_{t_r}$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{\mathscr{r}}_r = -\vec{\nabla}(t_r) \times \vec{v} = \vec{v} \times \vec{\nabla}(t_r)$$

and this proves A.

To prove Ⓑ: $(\vec{\mathscr{r}}_r \cdot \vec{\nabla})\,\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r \cdot \vec{\nabla}\,(\vec{r} - \vec{w}(t_r))$.

Then for any vector $\vec{a}$ it is easily shown that $(\vec{a} \cdot \vec{\nabla})\vec{r} = \vec{a}$.

Thus
$$(\vec{\mathscr{r}}_r \cdot \vec{\nabla})\,\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r - \vec{\mathscr{r}}_r \cdot \vec{\nabla}\,(\vec{w}(t_r))$$

Now
$$\vec{b} \cdot \vec{\nabla}\,(\vec{w}(t_r)) = b_x \frac{\partial}{\partial x}\vec{w}(t_r) + b_y \frac{\partial}{\partial y}\vec{w}(t_r) + b_z \frac{\partial}{\partial z}\vec{w}(t_r)$$
$$= b_x \left.\frac{d\vec{w}}{dt}\right|_{t_r}\frac{\partial t_r}{\partial x} + b_y \left.\frac{d\vec{w}}{dt}\right|_{t_r}\frac{\partial t_r}{\partial y} + \cdots$$
$$= \frac{d\vec{w}}{dt}\left[\vec{b} \cdot \vec{\nabla}(t_r)\right].$$

Thus.

$$(\vec{\mathscr{r}}_r \cdot \vec{\nabla})\,\vec{\mathscr{r}}_r = \vec{\mathscr{r}}_r - \left.\frac{d\vec{W}}{dt}\right|_{t_r} \vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r)$$

$$= \vec{\mathscr{r}}_r - \vec{v}\,(\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r))$$

which proves B. So

$$-c\,\vec{\nabla}(t_r) = \frac{1}{\mathscr{r}_r}\left\{ \vec{\mathscr{r}}_r \times \left[\vec{\nabla} \times \vec{\nabla}(t_r)\right] + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\,\vec{\mathscr{r}}_r \right\}.$$

$$= \frac{1}{\mathscr{r}_r}\left\{ \cancel{\vec{v}\,(\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r))} - \vec{\nabla}(t_r)(\vec{v} \cdot \vec{\mathscr{r}}_r) + \vec{\mathscr{r}}_r - \cancel{\vec{v}\,(\vec{\mathscr{r}}_r \cdot \vec{\nabla}(t_r))} \right\}$$

$$\Rightarrow c\,\vec{\nabla}(t_r) = \frac{1}{\mathscr{r}_r}\left\{ \vec{\nabla}(t_r)(\vec{v} \cdot \vec{\mathscr{r}}_r) - \vec{\mathscr{r}}_r \right\}.$$

$$\Rightarrow \vec{\nabla}(t_r)\left[\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r\right] = -\vec{\mathscr{r}}_r$$

$$\Rightarrow \vec{\nabla}(t_r) = \frac{-\vec{\mathscr{r}}_r}{\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r} \qquad \blacksquare$$

Lemma 2 $\quad \vec{\nabla}(\vec{\mathscr{r}}_r \cdot \vec{v}) = \vec{v} + (\vec{\mathscr{r}}_r \cdot \vec{a} - v^2)\vec{\nabla}(t_r)$

where

$$\vec{a} = \frac{d^2\vec{w}}{dt^2}\Big|_{t_r}.$$

Proof. $\quad \vec{\nabla}(\vec{\mathscr{r}}_r \cdot \vec{v}) = \vec{\mathscr{r}}_r \times (\vec{\nabla} \times \vec{v}) + \vec{v} \times (\vec{\nabla} \times \vec{\mathscr{r}}_r)$

evaluated at t_r $\quad + (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{v} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{\mathscr{r}}_r$

We saw in lemma 1 that

$$\vec{\nabla} \times \vec{\mathscr{r}}_r = \vec{v} \times \vec{\nabla}(t_r)$$

Then $\quad \vec{\nabla} \times \vec{v} = \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right)\hat{x} + \cdots$

$$= \left(\frac{dv_y}{dt}\Big|_{t_r} \frac{\partial t_r}{\partial z} - \frac{dv_z}{dt}\Big|_{t_r} \frac{\partial t_r}{\partial y}\right)\hat{x} + \cdots$$

$$= \left[a_y (\vec{\nabla} t_r)_z - a_z (\vec{\nabla} t_r)_y\right]\hat{x} + \cdots$$

$$= \vec{\nabla}(t_r) \times \vec{a}$$

So $\quad \vec{\nabla}(\vec{\mathscr{r}}_r \cdot \vec{v}) = \vec{\mathscr{r}}_r \times [\vec{\nabla}(t_r) \times \vec{a}] + \vec{v} \times [\vec{v} \times \vec{\nabla}(t_r)]$

$$+ (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{v} + (\vec{v} \cdot \vec{\nabla}) \cdot \vec{\mathscr{r}}_r$$

Then $\quad (\vec{\mathscr{r}}_r \cdot \vec{\nabla})\vec{v} = \mathscr{r}_{rx} \frac{\partial}{\partial x}\vec{v} + \mathscr{r}_{ry}\frac{\partial}{\partial y}\vec{v} + \cdots$

$$= \mathscr{r}_{rx} \frac{d\vec{v}}{dt}\Big|_{t_r} \frac{dt_r}{dx} + \cdots$$

$$= [\vec{\mathscr{r}} \cdot \vec{\nabla}(t_r)]\,\vec{a}$$

Also: $(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}}_r = v_x \frac{\partial}{\partial x}\vec{\mathscr{r}}_r + v_y \frac{\partial}{\partial y}\vec{\mathscr{r}}_r + \dots$ and $\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$

$$= v_x \hat{x} - [\vec{v}\cdot\vec{\nabla}(t_r)]\, v_x \hat{x} + \dots$$

$$= \vec{v} - [\vec{v}\cdot\vec{\nabla}(t_r)]\,\vec{v}$$

Thus: $\vec{\nabla}(\vec{\mathscr{r}}_r\cdot\vec{v}) = [\vec{\nabla}(t_r)]\,\vec{\mathscr{r}}_r\cdot\vec{a} - \cancel{\vec{a}\,[\vec{\mathscr{r}}_r\cdot\vec{\nabla}(t_r)]}$

$$+ \cancel{\vec{v}\,[\vec{v}\cdot\vec{\nabla}(t_r)]} - \vec{\nabla}(t_r)\, v^2$$

$$+ \cancel{\vec{a}\,[\vec{\mathscr{r}}_r\cdot\vec{\nabla}(t_r)]} + \vec{v} - \cancel{\vec{v}\,[\vec{v}\cdot\vec{\nabla}(t_r)]}$$

$$= \vec{v} + \vec{\nabla}(t_r)\left[\vec{\mathscr{r}}_r\cdot\vec{a} - v^2\right]$$

This proves the result. ⊠

Lemma 3 $\quad c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$

$$\Rightarrow \quad c^2(t-t_r)^2 = r^2 + w^2(t_r) - 2\vec{r}\cdot\vec{w}(t_r)$$

Differentiate w.r.t. t $\Rightarrow$ $c^2 \cancel{2}(t-t_r)\left[1 - \frac{\partial t_r}{\partial t}\right]$

$$= \cancel{2} w \left.\frac{dw}{dt}\right|_{t_r} \frac{\partial t_r}{\partial t} - \cancel{2}\vec{r}\cdot \left.\frac{d\vec{w}}{dt}\right|_{t_r} \frac{\partial t_r}{\partial t}$$

$$\Rightarrow \quad c^2(t-t_r) = \frac{\partial t_r}{\partial t}\left[c^2(t-t_r) + \vec{w}\cdot\vec{v} - \vec{r}\cdot\vec{v}\right]$$

$$\Rightarrow \quad c\,\mathscr{r}_r = \frac{\partial t_r}{\partial t}\left[c^2(t-t_r) - \vec{v}\cdot\vec{\mathscr{r}}_r\right]$$

$$\Rightarrow \quad \frac{\partial t_r}{\partial t} = \frac{c\,\mathscr{r}_r}{c\,\mathscr{r}_r - \vec{v}\cdot\vec{\mathscr{r}}_r} \qquad \boxtimes$$

Lemma 4 $\quad \frac{\partial}{\partial t}(\vec{v}\cdot\vec{\mathscr{r}}_r) = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\underbrace{\frac{\partial \vec{v}}{\partial t_r}}_{\vec{a}}\frac{\partial t_r}{\partial t}}\cdot\vec{\mathscr{r}}_r + \vec{v}\cdot\frac{\partial \vec{\mathscr{r}}_r}{\partial t}$

$$= \vec{a}\cdot\vec{\mathscr{r}}_r \frac{\partial t}{\partial t_r} + \vec{v}\left(-\frac{\partial \vec{w}}{\partial t}\right)$$

$$= \left(\vec{a}\cdot\vec{\mathscr{r}} - \vec{v}\cdot\vec{v}\right)\frac{\partial t}{\partial t_r}$$

$$= \left(\vec{\mathscr{r}}_r\cdot\vec{a} - v^2\right)\frac{\partial t}{\partial t_r} \qquad \boxtimes$$

Electric field

There are two contributions to $\vec{E}$:

a) $-\vec{\nabla} V$

b) $-\frac{\partial \vec{A}}{\partial t}$

Consider the scalar potential

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \vec{\nabla} \left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right)^{-1}$$

$$= -\frac{qc}{4\pi\epsilon_0} \left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right)^{-2} \left[c \vec{\nabla} \mathscr{r}_r - \vec{\nabla} \left(\vec{v} \cdot \vec{\mathscr{r}}_r \right) \right]$$

Now $\mathscr{r}_r = c(t - t_r) \Rightarrow \vec{\nabla} \mathscr{r}_r = -c \vec{\nabla} t_r$ and

$$\vec{\nabla} \left(\vec{v} \cdot \vec{\mathscr{r}}_r \right) = \vec{v} + \left(\vec{\mathscr{r}}_r \cdot \vec{a} - v^2 \right) \vec{\nabla} t_r$$

give

$$\vec{\nabla} V = \frac{-qc}{4\pi\epsilon_0} \frac{1}{\left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right)^2} \left[\left(v^2 - c^2 \right) \vec{\nabla} t_r - \vec{v} - \vec{\mathscr{r}}_r \cdot \vec{a} \, \vec{\nabla} t_r \right]$$

Then by Lemma 1

$$\vec{\nabla} V = \frac{-qc}{4\pi\epsilon_0} \frac{1}{\left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right)^2} \left[\frac{c^2 - v^2 + \vec{\mathscr{r}}_r \cdot \vec{a}}{\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r} \vec{\mathscr{r}}_r - \vec{v} \right]$$

$$= \frac{-qc}{4\pi\epsilon_0} \frac{1}{\left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right)^3} \left[\vec{\mathscr{r}}_r \left(c^2 - v^2 + \vec{\mathscr{r}}_r \cdot \vec{a} \right) - \vec{v} \left(\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r \right) \right]$$

Now consider the vector potential.

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 q c}{4\pi} \frac{\partial}{\partial t} \frac{\vec{v}(t_r)}{\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v}} = \frac{\mu_0 q c}{4\pi} \frac{\frac{\partial \vec{v}(t_r)}{\partial t} \left(\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v} \right) - \vec{v}(t_r) \frac{\partial}{\partial t} \left(\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v} \right)}{\left(\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v} \right)^2}$$

Then

$$\frac{\partial \vec v}{\partial t} = \frac{\partial \vec v}{\partial t_r}\frac{\partial t_r}{\partial t} = \vec a \,\frac{\partial t_r}{\partial t}$$

evaluated at t_r $= \frac{d\vec v}{dt}\Big|_{t_r}$

and

$$\frac{\partial(\mathscr{r}_r c)}{\partial t} = c\,\frac{\partial \mathscr{r}_r}{\partial t} = c\,\frac{\partial}{\partial t}\, c(t-t_r) = c^2\left[1-\frac{\partial t_r}{\partial t}\right]$$

$$\frac{\partial}{\partial t}(\vec{\mathscr{r}}_r\cdot\vec v) = \frac{\partial t_r}{\partial t}\,(\vec{\mathscr{r}}_r\cdot\vec a - v^2)$$

Thus

$$\frac{\partial \vec A}{\partial t} = \frac{\mu_0 q c}{4\pi}\,\frac{\vec a\left[\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v\right]\frac{\partial t_r}{\partial t} - \vec v\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right) - \frac{\partial t_r}{\partial t}\,(\vec{\mathscr{r}}_r\cdot\vec a - v^2)\right]}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)^2}$$

$$= \frac{\mu_0 q c}{4\pi}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)^2}\left\{\left[(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)\vec a + \vec v c^2 + (\vec{\mathscr{r}}_r\cdot\vec a)\vec v + \vec v v^2\right]\frac{\partial t_r}{\partial t} - c^2\vec v\right\}$$

$$= \frac{\mu_0 q c}{4\pi}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)^2}\left\{\frac{\mathscr{r}_r c}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)}\left[(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)\vec a + \vec v(c^2-v^2) + \vec v(\vec{\mathscr{r}}_r\cdot\vec a)\right] - c^2\vec v\right\}$$

$$= \frac{\mu_0 q c}{4\pi}\,\frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)^3}\left\{\mathscr{r}_r c\left[(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)\vec a + \vec v(c^2-v^2) + \vec v(\vec{\mathscr{r}}_r\cdot\vec a)\right] - c^2\vec v(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)\right\}$$

$\mu_0 = \frac{1}{c^2\epsilon_0}$

$$= \frac{q}{4\pi\epsilon_0 c}\,\frac{1}{(\dots)^3}\left\{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)(\mathscr{r}_r c\,\vec a - c^2\vec v) + \mathscr{r}_r c\,\vec v\,(c^2 - v^2 + \vec{\mathscr{r}}_r\cdot\vec a)\right\}$$

$$= \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\dots)^3}\left\{(\mathscr{r}_r c - \vec{\mathscr{r}}_r\cdot\vec v)\left(\frac{\mathscr{r}_r\vec a}{c} - \vec v\right) + \frac{\mathscr{r}_r}{c}\vec v\,(c^2 - v^2 + \vec{\mathscr{r}}_r\cdot\vec a)\right\}$$

$$\Rightarrow\quad \vec\nabla V + \frac{\partial \vec A}{\partial t} = \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\dots)^3}\left[-\vec{\mathscr{r}}_r(c^2 - v^2 + \vec{\mathscr{r}}_r\cdot\vec a) + (\mathscr{r}_r c - \vec v\cdot\vec{\mathscr{r}}_r)\frac{\mathscr{r}_r\vec a}{c} + \frac{\mathscr{r}_r}{c}\vec v\,(c^2 - v^2 + \vec{\mathscr{r}}_r\cdot\vec a)\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\,\frac{1}{(\dots)^3}\left\{(\mathscr{r}_r c - \vec v\cdot\vec{\mathscr{r}}_r)\frac{\mathscr{r}_r\vec a}{c} + (c^2 - v^2 + \vec{\mathscr{r}}_r\cdot\vec a)\left[\frac{\mathscr{r}_r}{c}\vec v - \vec{\mathscr{r}}_r\right]\right\}$$

$$\Rightarrow \quad \vec{\nabla} V + \frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{(\ldots)^3} \left\{ (\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r)\, \mathscr{r}_r \vec{a} + (c^2 - v^2 + \vec{\mathscr{r}}_r \cdot \vec{a})(\mathscr{r}_r \vec{v} - c\vec{\mathscr{r}}_r) \right]$$

$$\Rightarrow \quad \vec{E} = \frac{-q}{4\pi\epsilon_0} \frac{1}{(\mathscr{r}_r c - \vec{\mathscr{r}}_r \cdot \vec{v})^3} \left[\mathscr{r}_r \vec{a}\, (\mathscr{r}_r c - \vec{v}\cdot\vec{\mathscr{r}}_r) + (c^2 - v^2 + \vec{\mathscr{r}}_r \cdot \vec{a})(\mathscr{r}_r \vec{v} - c\vec{\mathscr{r}}_r) \right]$$

Now define

$$\vec{u} := c\hat{\mathscr{r}}_r - \vec{v}$$

Then

$$E = -\frac{q}{4\pi\epsilon_0} \frac{1}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3} \left\{ \mathscr{r}_r \vec{a}\, (\vec{u}\cdot\vec{\mathscr{r}}_r) - (c^2 - v^2 + \vec{\mathscr{r}}_r \cdot \vec{a})\, \mathscr{r}_r \vec{u} \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3} \left\{ (c^2 - v^2)\mathscr{r}_r \vec{u} + \mathscr{r}_r \left[\vec{u}\,(\vec{u}\cdot\vec{\mathscr{r}}_r) - \mathscr{r}_r \vec{u}\,(\vec{\mathscr{r}}_r \cdot \vec{a}) \right] \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3} \left\{ (c^2 - v^2)\, \vec{u} - \vec{\mathscr{r}}_r \times (\vec{a} \times \vec{u}) \right\}.$$

So

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}_r}{(\vec{u}\cdot\vec{\mathscr{r}}_r)^3} \left\{ (c^2 - v^2)\vec{u} + \vec{\mathscr{r}}_r \times (\vec{u} \times \vec{a}) \right\}.$$

A similar vector calculus calculation gives

$$\vec{B} = \frac{1}{c} (\hat{\mathscr{r}}_r \times \vec{E})$$

Thus the scheme is:

Given $\vec{w}(t)$ $\rightarrow$ ① Determine retarded time via

$$c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

② Determine retarded separation vector

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r)$$

$\rightarrow$ ④ Determine

$$\vec{v} = \left.\frac{d\vec{w}}{dt}\right|_{t_r} \quad \text{and} \quad \vec{a} = \left.\frac{d^2\vec{w}}{dt^2}\right|_{t_r}$$

⑤ Determine

$$\vec{u} = c\hat{\mathscr{r}}_r - \vec{v}$$

The fields are

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}_r}{(\vec{\mathscr{r}}_r\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}_r\times(\vec{u}\times\vec{a})\right]}$$

$$\boxed{\vec{B} = \frac{1}{c}\hat{\mathscr{r}}_r\times\vec{E}}$$

1 Fields produced by a charge moving with constant velocity

Consider a charged particle moving with constant velocity. Thus

$$\mathbf{w}(t) = \mathbf{v}\,t$$

where $\mathbf{v}$ is the velocity.

a) Show that

$$\mathbf{u} = \frac{\mathbf{r} - \mathbf{v}\,t}{t - t_r}.$$

b) Determine an expression for the electric field.

c) Determine an expression for the magnetic field.

d) Let $\mathbf{R} = \mathbf{r} - \mathbf{v}t$. Provide a geometric interpretation of this.

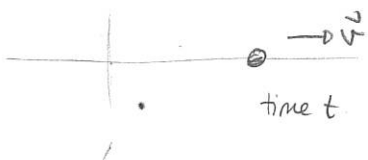
e) Express the electric field in terms of $\mathbf{R}$. Note that it can be shown that

$$1 - \frac{\boldsymbol{\mathscr{r}}_r \cdot \mathbf{v}}{c} = \frac{R}{\mathscr{r}_r}\sqrt{1 - \frac{v^2}{c^2}\sin^2\theta}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{v}$.

f) Show that the magnitude of the Poynting vector scales as $1/R^4$. If you were to consider the energy that flows out of a very large sphere centered at the origin, what would this imply?

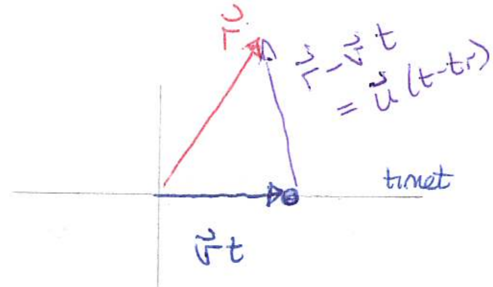
Answer a)



$$\vec{u} = c\,\hat{\mathscr{r}}_r - \vec{v} = \frac{c}{\mathscr{r}_r}\vec{\mathscr{r}}_r - \vec{v}$$

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}\,t_r$$

$$\Rightarrow \vec{u} = \frac{c}{\mathscr{r}_r}(\vec{r} - \vec{v}\,t_r) - \vec{v}$$



But $\mathscr{r}_r = c(t - t_r)$

$$\Rightarrow \vec{u} = \frac{\vec{r} - \vec{v}\,t_r}{t - t_r} - \vec{v} = \frac{\vec{r} - \vec{v}\,t_r - \vec{v}(t - t_r)}{t - t_r}$$

$$\Rightarrow \vec{u} = \frac{\vec{r} - \vec{v}\,t}{t - t_r}$$

b) Here $\vec{a} = 0$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\eta_r}{(\vec{u}\cdot\vec{\eta}_r)^3} (c^2 - v^2)\vec{u}$$

Then $\eta_r = c(t - t_r)$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c\,(c^2-v^2)}{(\vec{u}\cdot\vec{\eta}_r)^3} \underbrace{(t-t_r)\,\vec{u}}_{\vec{r}-\vec{v}t}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{c\,(c^2-v^2)}{(\vec{u}\cdot\vec{\eta}_r)^3} (\vec{r}-\vec{v}t)$$

Then
$$\vec{u}\cdot\vec{\eta}_r = (c\hat{\eta}_r - \vec{v})\cdot\vec{\eta}_r$$
$$= (c\eta_r - \vec{v}\cdot\vec{\eta}_r) = c\eta_r(1 - \vec{v}\cdot\vec{\eta}_r / c\eta_r)$$

$$\Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{c\,(c^2-v^2)}{c^3\eta_r^3(1 - \vec{v}\cdot\hat{\eta}_r/c)^3} (\vec{r}-\vec{v}t)$$

$$= \frac{q}{4\pi\epsilon_0}\left(1 - \frac{v^2}{c^2}\right)\frac{1}{\eta_r^3}\frac{1}{(1-\hat{\eta}_r\cdot\vec{v}/c)^3}(\vec{r}-\vec{v}t)$$

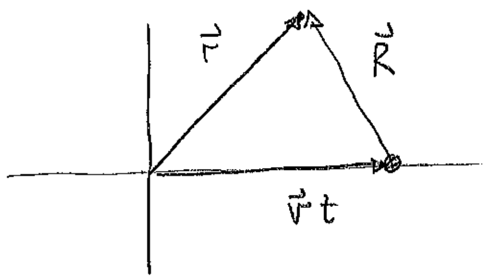
c) $\vec{B} = \frac{1}{c}\hat{\eta}_r \times \vec{E} = \frac{1}{c}\frac{1}{\eta_r}\vec{\eta}_r \times \vec{E}$

Then $\vec{\eta}_r = \vec{r} - \vec{v}t_r$ and
$$\vec{\eta}_r \times (\vec{r}-\vec{v}t)$$
$$= (\vec{r}-\vec{v}t_r)\times(\vec{r}-\vec{v}t)$$
$$= \vec{v}\times\vec{r}\,(t - t_r)$$
$$= \vec{v}\times\vec{r}\,(+\eta_r/c)$$

Thus

$$\vec{B} = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{1}{\mathscr{r}^3}\frac{1}{\left(1-\hat{\mathscr{r}}\cdot\vec{v}/c\right)^3}\,\vec{v}\times\vec{r}$$

d)



vector from current location to field point

e) $$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(1-\frac{v^2}{c^2}\right)\frac{1}{\mathscr{r}^3}\frac{\vec{R}}{R^3/\mathscr{r}^3\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2}\frac{\left(1-v^2/c^2\right)}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\hat{R}}$$

f) $$\vec{B} = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{1}{R^3}\frac{1}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\,\vec{v}\times\vec{r}$$

Then $\vec{v}\times\vec{r} = \vec{v}\times(\vec{R}-\vec{v}t) = \vec{v}\times\vec{R}$

$$\Rightarrow \vec{B} = \frac{q}{4\pi\epsilon_0}\frac{1}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{1}{R^3}\frac{1}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\,\vec{v}\times\vec{R}$$

$$\boxed{\vec{B} = \frac{1}{c^2}\frac{q}{4\pi\epsilon_0}\frac{1}{R^2}\left(1-\frac{v^2}{c^2}\right)\frac{1}{\left(1-\frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}\,\vec{v}\times\hat{R}}$$

Note that $\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$

Then

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \sim \frac{1}{c^2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{R^4} \cdots$$

For a sphere of radius r centered at the origin, energy flow is

$$\oint_{\text{surface}} \vec{S}\cdot d\vec{a} \qquad \frac{1}{R^4} \qquad r^2$$

the integral $\sim \frac{1}{r^2}$

For very large r this is approximately zero

Note that the fields appear as