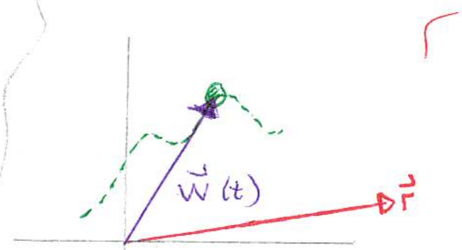
Tues: HW 5pm ~~Tues~~ Thurs Read 1.3.2.

Fri: HW 5pm

Potentials produced by a moving point charge

The process for determining potentials produced by a moving point particle with charge q is:

Specify particle trajectory



position vector $\vec{w}(t)$

In the Lorentz gauge the potentials satisfy

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

Retarded potential solutions

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\mathcal{R}}\, d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathcal{R}}\, d\tau'$$

where $\vec{\mathcal{R}} = \vec{r} - \vec{r}\,'$

$t_r = t - \mathcal{R}/c$

Charge density

$$\rho(\vec{r}\,',t) = q\,\delta(\vec{r}\,' - \vec{w}(t))$$

current density

$$\vec{J}(\vec{r}\,',t) = \vec{v}(t)\, q\,\delta(\vec{r}\,' - \vec{w}(t))$$

where $\vec{v} = \dfrac{d\vec{w}}{dt}$

Determine the retarded time by solving

$$c(t - t_r) = |\vec{r} - \vec{w}(t_r)|$$

Determine retarded position

$$\vec{r}_r = \vec{w}(t_r)$$

Determine retarded separation vector

$$\vec{\mathcal{R}}_r = \vec{r} - \vec{r}_r$$

Scalar potential $V(\vec{r},t) = \dfrac{qc}{4\pi\epsilon_0}\,\dfrac{1}{\mathcal{R}_r c - \vec{v}\cdot\vec{\mathcal{R}}_r}$

Vector potential $\vec{A}(\vec{r},t) = \dfrac{\mu_0 qc}{4\pi}\,\dfrac{\vec{v}(t_r)}{\mathcal{R}_r c - \vec{v}\cdot\vec{\mathcal{R}}_r}$

The last set of potentials are called the Liénard-Wichert potentials. Note that

$$\frac{1}{\mu_0 \epsilon_0} = c^2 \Rightarrow \mu_0 = \frac{1}{c^2 \epsilon_0}$$

$$\Rightarrow \vec{A}(\vec{r},t) = \frac{qc}{4\pi\epsilon_0} \frac{\vec{v}(t_r)/c^2}{\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r} = \frac{\vec{v}(t_r)}{c^2} V(\vec{r},t).$$

Thus we get the Liénard-Wichert potentials as:

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{qc}{4\pi\epsilon_0} \frac{1}{\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r} \\ \vec{A}(\vec{r},t) &= \frac{\vec{v}(t_r)}{c^2} V(\vec{r},t) = \frac{\mu_0 q c}{4\pi} \frac{\vec{v}(t_r)}{\mathscr{r}_r c - \vec{v} \cdot \vec{\mathscr{r}}_r} \end{aligned}}$$

The fields are then

$$\boxed{\begin{aligned} \vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}}$$

1 Retardation for a particle traveling with constant velocity

A charged particle travels with constant speed v along the $+x$ axis, passing the origin at $t = 0$. Suppose that one wants to determine the scalar potential at $\mathbf{r} = y\hat{\mathbf{y}}$ at time $t > 0$.

a) Determine an expression for the retarded time.

b) Suppose that $v = c/\sqrt{2}$. Determine an expression for the retarded position.

c) Suppose that $v = c/\sqrt{2}$. Determine an expression for the magnitude of the retarded separation vector.

Answer: a) $\vec{w}(t) = vt\hat{x}$ and $\vec{r} = y\hat{y}$

Solve for t_r

$$t_r = t - \frac{\mathscr{r}}{c} \Rightarrow c(t - t_r) = \mathscr{r} \Rightarrow c(t-t_r) = |\vec{r} - \vec{w}(t_r)|$$

$$\Rightarrow c(t-t_r) = |y\hat{y} - vt_r\hat{x}|$$

$$\Rightarrow c^2(t-t_r)^2 = y^2 + v^2t_r^2$$

$$\Rightarrow c^2t_r^2 - v^2t_r^2 - 2c^2tt_r + c^2t^2 - y^2 = 0$$

$$\Rightarrow t_r^2(c^2-v^2) + t_r(-2c^2t) + c^2t^2 - y^2 = 0$$

$$\Rightarrow t_r = \frac{2c^2t \pm \sqrt{4c^4t^2 - 4(c^2-v^2)(c^2t^2-y^2)}}{2(c^2-v^2)}$$

$$= \frac{c^2t \pm \sqrt{\cancel{c^4t^2} - \cancel{c^4t^2} + v^2c^2t^2 - v^2y^2 + c^2y^2}}{c^2-v^2}$$

$$= \frac{c^2t \pm \sqrt{c^2v^2t^2 + y^2(c^2-v^2)}}{c^2-v^2}$$

$$= \frac{c^2}{c^2-v^2}\left[t \pm \frac{1}{c^2}\sqrt{}\right]$$

$$\Rightarrow \quad t_r = \frac{1}{1-v^2/c^2}\left[\, t \pm \sqrt{t^2 \, v^2/c^2 + \frac{y^2}{c^2}\left(1 - v^2/c^2\right)}\,\right]$$

We need that $t > t_r$ and the $+$ sign would violate this.

So

$$t_r = \frac{1}{1-v^2/c^2}\left[\, t - \sqrt{t^2 \, v^2/c^2 + y^2/c^2 \left(1 - v^2/c^2\right)}\,\right]$$

b) In this case

$$t_r = \frac{1}{1-1/2}\left[\, t - \sqrt{\tfrac{1}{2} t^2 + y^2/c^2 \, \tfrac{1}{2}}\,\right]$$

$$= 2\left[\, t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\,\right]$$

$$= 2t\left[\, 1 - \frac{1}{\sqrt{2}}\sqrt{1 + y^2/c^2t^2}\,\right]$$

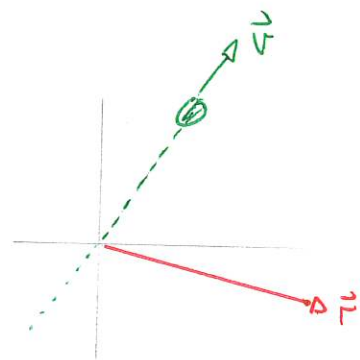
$$\vec{r}_r = \vec{w}(t_r) = 2vt\left[\, 1 - \frac{1}{\sqrt{2}}\sqrt{1 + y^2/c^2t^2}\,\right]\hat{x}$$

c) $$\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r = y\hat{y} - 2vt\,[\;\ldots\;]\,\hat{x}$$

$$\mathscr{r}_r = \sqrt{y^2 + 4v^2t^2\left[1 - \frac{1}{\sqrt{2}}\sqrt{1 + y^2/c^2t^2}\right]^2}$$

Particle moving with constant velocity

The simplest non-trivial situation is a particle moving with constant velocity. Suppose that



* the particle passes through the origin at $t=0$
* " " velocity is $\vec{v}$

we aim to determine the field at any point $\vec{r}$. Then

$$\vec{w} = \vec{v}\,t$$

We show that

> For the field at location $\vec{r}$ at time t, the retarded time is
> $$t_r = \frac{c^2 t - \vec{r}\cdot\vec{v} - \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

Derivation

$$c\,|t - t_r| = |\vec{r} - \vec{w}(t_r)| = |\vec{r} - \vec{v}\,t_r|$$

$$\Rightarrow c^2 (t - t_r)^2 = (\vec{r} - \vec{v}\,t_r)\cdot(\vec{r} - \vec{v}\,t_r) = r^2 + v^2 t_r^2 - 2\vec{r}\cdot\vec{v}\,t_r$$

$$\Rightarrow c^2 (t^2 - 2 t t_r + t_r^2) = r^2 + v^2 t_r^2 - 2\vec{r}\cdot\vec{v}\,t_r$$

$$\Rightarrow t_r^2 (c^2 - v^2) + t_r(-2tc^2 + 2\vec{r}\cdot\vec{v}) + c^2 t^2 - r^2 = 0$$

$$\Rightarrow t_r = \frac{2tc^2 - 2\vec{r}\cdot\vec{v} \pm \sqrt{4\,(tc^2 - \vec{r}\cdot\vec{v})^2 - 4(c^2 - v^2)(c^2 t^2 - r^2)}}{2(c^2 - v^2)}$$

$$\Rightarrow t_r = \frac{(c^2 t - \vec{r}\cdot\vec{v}) \pm \sqrt{(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}$$

$$= \frac{c^2 t - \vec{r}\cdot\vec{v} \pm \sqrt{\cancel{c^4 t^2} - 2c^2 t\,\vec{r}\cdot\vec{v} + (\vec{r}\cdot\vec{v})^2 - \cancel{c^4 t^2} + (c^2 - v^2)\,r^2 + c^2 v^2 t^2}}{c^2 - v^2}$$

$$= \frac{c^2 t - \vec{r}\cdot\vec{v} \pm \sqrt{(\vec{r}\cdot\vec{v})(\vec{r}\cdot\vec{v} - 2c^2 t) + c^2 v^2 t^2 + (c^2 - v^2)\,r^2}}{c^2 - v^2}$$

$$\Rightarrow t_r = \frac{1}{1 - v^2/c^2}\left[t - \frac{\vec{r}\cdot\vec{v}}{c^2} \pm \sqrt{\frac{v^2}{c^2} t + \left(1 - \frac{v^2}{c^2}\right) r^2/c^2 + \frac{\vec{r}\cdot\vec{v}}{c^2}\left(\frac{\vec{r}\cdot\vec{v}}{c^2} - 2t\right)}\,\right]$$

We need the − sign for $\vec{r}\cdot\vec{v} = 0$ and thus here too ∎

2 Retarded separation for a particle moving with constant velocity

Consider a point source particle with charge q and which moves with constant velocity $\mathbf{v}$, passing through the origin at $t = 0$.

a) Determine an expression for the retarded separation vector and the retarded separation distance.

b) Show that

$$\mathcal{R}_r c - \mathbf{v} \cdot \boldsymbol{\mathcal{R}}_r = \sqrt{(c^2 t - \mathbf{v} \cdot \mathbf{r})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}$$

Answer. a) $\vec{\mathcal{R}}_r = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{v}\, t_r$

$\mathcal{R}_r = |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$

b) $\mathcal{R}_r c = c^2(t - t_r)$

$\vec{v} \cdot \vec{\mathcal{R}}_r = \vec{v} \cdot \vec{r} - v^2 t_r.$

$$\Rightarrow \quad \mathcal{R}_r c - \vec{v} \cdot \vec{\mathcal{R}}_r = c^2(t - t_r) + v^2 t_r - \vec{v} \cdot \vec{r}$$

$$= c^2 t - \vec{v} \cdot \vec{r} - (c^2 - v^2) t_r$$

$$= c^2 t - \vec{v} \cdot \vec{r} - c^2 t + \vec{v} \cdot \vec{r} + \sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)} \quad \blacksquare$$

Thus for a point source moving with constant velocity

$$\boxed{V(\vec{r}, t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{\sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}}$$

$$\boxed{\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\, \vec{v}}{\sqrt{(c^2 t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}}$$

3 Potentials for a particle moving with constant velocity.

A charged particle travels with constant velocity along the $+x$ axis, passing the origin at $t = 0$.

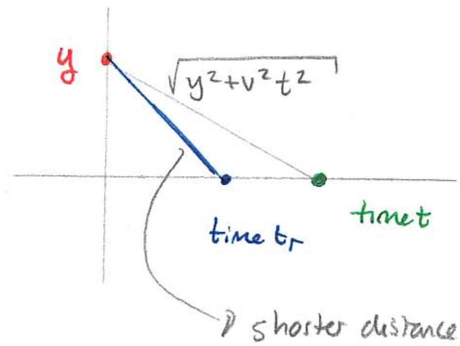
a) Determine the scalar and vector potentials at any point along the y axis.

b) Determine the potentials for the case where the velocity is zero. Are these consistent with those obtained by electrostatics and magnetostatics?

c) Determine the potentials for the case where $v \ll c$.

Answer: a)

$$\vec{v} = v\hat{x}$$
$$\vec{r} = y\hat{y}$$



$$\vec{v}\cdot\vec{r} = 0$$

$$\Rightarrow V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{\left[c^4t^2 + (c^2-v^2)(r^2-c^2t^2)\right]^{1/2}}$$

$$= \frac{qc}{4\pi\epsilon_0} \frac{1}{\left[\cancel{c^4t^2} + c^2r^2 - v^2r^2 + c^2v^2t^2 - \cancel{c^4t^2}\right]^{1/2}}$$

$$= \frac{qc}{4\pi\epsilon_0} \frac{1}{\left[c^2(1-v^2/c^2)r^2 + c^2v^2t^2\right]^{1/2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(1-v^2/c^2)r^2 + v^2t^2}}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(1-v^2/c^2)y^2 + v^2t^2}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \frac{qv\hat{x}}{\sqrt{(1-v^2/c^2)y^2 + v^2t^2}}$$

b) $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0 y}$ ✓ $\qquad \vec{A} = 0$ ✓

c) $V(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0} \dfrac{1}{\sqrt{y^2+v^2t^2}}$ $\qquad \vec{A}(\vec{r},t) = \dfrac{\mu_0}{4\pi} \dfrac{q\vec{v}}{\sqrt{y^2+v^2t^2}}$

distance to current particle location (effectively not retarded)

Fields for a particle moving with constant velocity

In order to calculate fields note that

$$V(\vec{r},t) = \frac{qc}{4\pi\epsilon_0} \frac{1}{f(\vec{r},t)^{1/2}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 c}{4\pi} q \frac{\vec{v}}{f(\vec{r},t)^{1/2}}$$

where $f(\vec{r},t) = (c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)$. Then

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_0} \vec{\nabla} f(\vec{r},t)^{-1/2}$$

$$= \frac{qc}{4\pi\epsilon_0}\left(-\frac{1}{2}\right) f(\vec{r},t)^{-3/2} \vec{\nabla} f(\vec{r},t)$$

$$= \frac{qc}{4\pi\epsilon_0}\left(-\frac{1}{2}\right) f(\vec{r},t)^{-3/2} \left[\vec{\nabla}(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)\vec{\nabla} r^2\right]$$

$$= \frac{qc}{4\pi\epsilon_0}\left(-\frac{1}{2}\right) f(\vec{r},t)^{-3/2} \left[2(c^2 t - \vec{r}\cdot\vec{v})(-\vec{v}) + (c^2-v^2)\, 2\vec{r}\right]$$

$$= +\frac{qc}{4\pi\epsilon_0} f(\vec{r},t)^{-3/2}\left[(c^2 t - \vec{r}\cdot\vec{v})\vec{v} + (c^2-v^2)\vec{r}\right]$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0 c\, q}{4\pi}\vec{v}\left(-\frac{1}{2}\right) f(\vec{r},t)^{-3/2}\frac{\partial f}{\partial t}$$

$$= \frac{\mu_0 c q}{4\pi}\vec{v}\left(-\frac{1}{2}\right) f(\vec{r},t)^{-3/2}\left[2(c^2 t - \vec{r}\cdot\vec{v})c^2 + (c^2-v^2)(-2c^2 t)\right]$$

$$= -\frac{\mu_0 c q}{4\pi}\vec{v} f(\vec{r},t)^{-3/2}\left[+c^2(c^2 t - \vec{r}\cdot\vec{v}) - (c^2-v^2)c^2 t\right]$$

$$= -\frac{\mu_0 c^3 q}{4\pi}\vec{v} f(\vec{r},t)^{-3/2}\left[\cancel{c^2 t} - \vec{r}\cdot\vec{v} - \cancel{c^2 t} + v^2 t\right] \qquad \epsilon_0\mu_0 = \frac{1}{c^2}$$

$$= -\frac{qc}{4\pi\epsilon_0}\vec{v} f(\vec{r},t)^{-3/2}\left[v^2 t - \vec{r}\cdot\vec{v}\right]$$

Then

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{qc}{4\pi\epsilon_0} f(\vec{r},t)^{-3/2}\left[(c^2 t - \vec{r}\cdot\vec{v})\vec{v} - (c^2 - v^2)\vec{r}\right]$$

$$+ \frac{qc}{4\pi\epsilon_0} f(\vec{r},t)^{-3/2}\,\vec{v}\left[v^2 t - \vec{r}\cdot\vec{v}\right]$$

$$= -\frac{qc}{4\pi\epsilon_0} f(\vec{r},t)^{-3/2}\left\{\vec{v}\,(c^2 t \cancel{- \vec{r}\cdot\vec{v}} \cancel{+ \vec{r}\cdot\vec{v}} - v^2 t) - (c^2 - v^2)\vec{r}\right.$$

$$= \frac{qc}{4\pi\epsilon_0} f(\vec{r},t)^{-3/2}\left\{(c^2 - v^2)(\vec{r} - \vec{v}t)\right\}.$$

$$= \frac{qc}{4\pi\epsilon_0}\frac{(c^2-v^2)(\vec{r}-\vec{v}t)}{\left[(c^2 t - \vec{r}\cdot\vec{v})^2 + (c^2-v^2)(r^2 - c^2t^2)\right]^{3/2}}$$

$$= \frac{qc^3}{4\pi\epsilon_0}\frac{(1-v^2/c^2)(\vec{r}-\vec{v}t)}{\left[(ct - \vec{r}\cdot\vec{v}/c)^2 + (1-v^2/c^2)(r^2 - c^2t^2)\right]^{3/2} c^3}$$

$$= \frac{q}{4\pi\epsilon_0}\left(1-\frac{v^2}{c^2}\right)\frac{\vec{r}-\vec{v}t}{\left[(ct - \vec{r}\cdot\vec{v}/c)^2 + (1-v^2/c^2)(r^2 - c^2t^2)\right]^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0}\left(1-v^2/c^2\right)\frac{\vec{r}-\vec{v}t}{\left[\cancel{c^2t^2} - 2\vec{r}\cdot\vec{v}t + (\vec{r}\cdot\vec{v})^2/c^2 + r^2 - \frac{v^2r^2}{c^2} - \cancel{c^2t^2} + v^2t^2\right]^{3/2}}$$

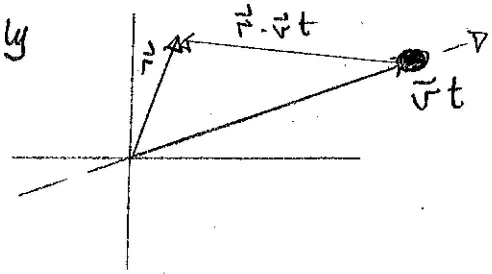
$$= \frac{q}{4\pi\epsilon_0}\left(1-v^2/c^2\right)\frac{\vec{r}-\vec{v}t}{\left[(\vec{r}-\vec{v}t)\cdot(\vec{r}-\vec{v}t) - \frac{v^2}{c^2}r^2 + \frac{(\vec{r}\cdot\vec{v})^2}{c^2}\right]^{3/2}}$$

Note that if $v \ll c$ this gives

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{(\vec{r}-\vec{v}t)}{|\vec{r}-\vec{v}t|^3}$$

This is the field produced by a point charge at $\vec{r}-\vec{v}t$

In this case the information from the particle propagates effectively instantaneously



Then the magnetic field is

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 c q}{4\pi}\,\vec{\nabla}\times\left[f^{-1/2}\,\vec{v}\right]$$

$$= \frac{\mu_0 c q}{4\pi}\left[f^{-1/2}\,\cancel{\vec{\nabla}\times\vec{v}} + \left(\vec{\nabla} f^{-1/2}\right)\times\vec{v}\right]$$

$$= \frac{\mu_0 c q}{4\pi} f(\vec{r},t)^{-3/2}\left[\left(c^2 t - \vec{r}\cdot\vec{v}\right)\vec{v} - \left(c^2-v^2\right)\vec{r}\right]\times\vec{v}$$

$$= -\frac{\mu_0 c q}{4\pi} f(\vec{r},t)^{-3/2}\left[\left(c^2-v^2\right)\vec{r}\times\vec{v}\right]$$

$$= \frac{\mu_0 c^3 q}{4\pi}\left(1-\frac{v^2}{c^2}\right)\frac{\vec{v}\times\vec{r}}{\left[\left(c^2 t-\vec{r}\cdot\vec{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0 c}\left(1-\frac{v^2}{c^2}\right)\frac{\vec{v}\times\vec{r}}{\left[\cancel{c^4t^2}+\left(\vec{r}\cdot\vec{v}\right)^2-2c^2t\,\vec{r}\cdot\vec{v}+c^2r^2-v^2r^2-\cancel{c^4t^2}+c^2v^2t^2\right]^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0 c^2}\left(1-v^2/c^2\right)\frac{\vec{v}\times\vec{r}}{\left[r^2-2t\,\vec{r}\cdot\vec{v}+v^2t^2-\frac{v^2}{c^2}r^2+\frac{\left(\vec{r}\cdot\vec{v}\right)^2}{c^2}\right]^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0 c^2}\left(1-v^2/c^2\right)\frac{\vec{v}\times\vec{r}}{\left[\left(\vec{r}-\vec{v}t\right)\cdot\left(\vec{r}-\vec{v}t\right)-\frac{v^2}{c^2}r^2+\frac{\left(\vec{r}\cdot\vec{v}\right)^2}{c^2}\right]^{3/2}}$$

This gives the direction of the magnetic field as perpendicular to $\vec{v}$ and $\vec{r}$

To simplify these let

$$\vec{R} = \vec{r} - \vec{v}t.$$

and consider

$$(\vec{r}\cdot\vec{v})^2 - v^2 r^2$$

which appears in the $[\ldots]^{3/2}$ term. Then

$$\vec{R}\cdot\vec{v} = \vec{r}\cdot\vec{v} - v^2 t \quad \Rightarrow (\vec{r}\cdot\vec{v}) = \vec{R}\cdot\vec{v} + v^2 t$$

$$\Rightarrow (\vec{r}\cdot\vec{v})^2 = (\vec{R}\cdot\vec{v})^2 + v^4 t^2 + 2v^2 t\,\vec{R}\cdot\vec{v}$$

$$\Rightarrow (\vec{r}\cdot\vec{v})^2 = R^2 v^2 \cos^2\theta + v^4 t^2 + 2v^2 t\,\vec{R}\cdot\vec{v}$$

where θ is the angle between $\vec{R}$ and $\vec{v}$. Thus

$$(\vec{r}\cdot\vec{v})^2 - r^2 v^2 = R^2 v^2 \cos^2\theta + v^2\left[v^2 t^2 + 2t\,\vec{R}\cdot\vec{v} - r^2\right]$$

$$= R^2 v^2 \cos^2\theta + v^2\left[v^2 t^2 + 2t\,\vec{r}\cdot\vec{v} - 2v^2 t^2 - r^2\right]$$

$$= R^2 v^2 \cos^2\theta - v^2\left[r^2 - 2t\,\vec{r}\cdot\vec{v} + v^2 t^2\right]$$

$$= R^2 v^2 \cos^2\theta - v^2 R^2$$

$$= R^2 v^2 (\cos^2\theta - 1) = -R^2 v^2 \sin^2\theta$$

Thus the term in the denominator is

$$(\vec{r}-\vec{v}t)(\vec{r}-vt) - R^2 v^2/c^2 \sin^2\theta = R^2\left(1 - v^2/c^2 \sin^2\theta\right)$$

The simplification is then:

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left(1-v^2/c^2\right)\frac{\vec{R}}{R^3\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}$$

For the magnetic field.

$$\vec{B} = \frac{q}{4\pi\epsilon_0 c^2}\,1/\left(1-v^2/c^2\right)\frac{\vec{v}\times\vec{r}}{R^3\left\{1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}}$$

But $\vec{v}\times\vec{r} = \vec{v}\times\vec{R}$ and this gives

$$\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

Thus

$$\boxed{\begin{array}{l}\text{For a particle moving with constant velocity } \vec{v} \\[1em] \vec{E}(\vec{r},t) = \dfrac{q}{4\pi\epsilon_0}\left(1-v^2/c^2\right)\dfrac{\vec{R}}{R^3\left[1-\frac{v^2}{c^2}\sin^2\theta\right]^{3/2}} \\[1.5em] \vec{B}(\vec{r},t) = \dfrac{1}{c^2}\vec{v}\times\vec{E} \\[1.5em] \text{where } \vec{R} = \vec{r}-\vec{v}t \text{ and } \theta \text{ is the angle between } \vec{R} \text{ and } \vec{v}\end{array}}$$