Phys 312 2025

Lecture 20

Fri: HW 5pm

Tues: HW 5pm

Tues: Read

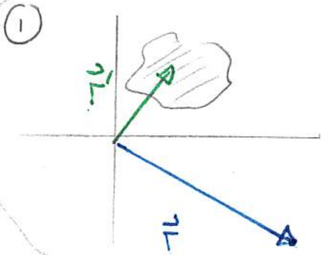
Retarded potentials

In general, the potentials produced by time-dependent source charge densities, $\rho(\vec{r}',t)$ and current densities $\vec{J}(\vec{r}',t)$ can be determined by working in the Lorentz gauge. Here

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t} = 0.$$

Then the retarded potentials appear as follows:

①



a) pick a field point $\vec{r}$ and time t

b) pick a point in the source distribution $\vec{r}'$

c) form the separation vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

and determine its magnitude $\mathscr{r} = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}$

② Form the retarded time

$$t_r = t - \mathscr{r}/c$$

This depends on the location within the source

$$t_r = t_r(\vec{r}, \vec{r}')$$

→ ③ a) The retarded scalar potential is

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{\mathscr{r}}\, d\tau'$$

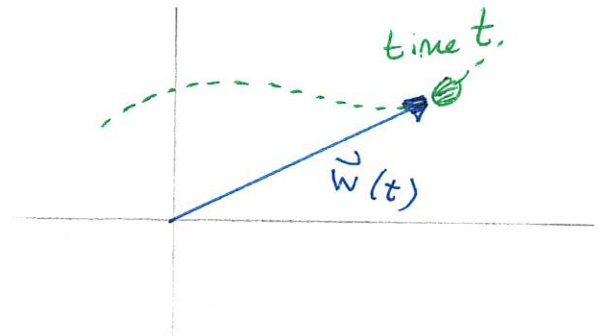
b) the retarded vector potential is

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{\mathscr{r}}\, d\tau'$$

Potentials produced by a moving point charge

Suppose that a point particle has fixed charge and moves along a known trajectory. We would like to determine the fields produced by this at any location. This requires:

1) the charge q
2) the position vector for the charge $\vec{w}(t)$



Then the source charge density is

$$\boxed{\rho(\vec{r},t) = q\,\delta^3(\vec{r} - \vec{w}(t))}$$

and the source current density is

$$\vec{J}(\vec{r},t) = \vec{v}\,\rho(\vec{r},t)$$

$$\Rightarrow \boxed{\vec{J}(\vec{r},t) = \frac{d\vec{w}}{dt}\, q\, \delta^3(\vec{r} - \vec{w}(t))}$$

We could substitute these into the retarded potentials.

Scalar potential

The retarded scalar potential would be

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{q}{\mathscr{r}}\,\delta^3(\vec{r}\,' - \vec{w}(t_r))\,d\tau' = \frac{1}{4\pi\epsilon_0}\int \frac{q}{|\vec{r} - \vec{r}\,'|}\,\delta(\vec{r}\,' - \vec{w}(t_r))\,d\tau'$$

It would appear that a simple substitution of $\vec{r}\,' = w(t_r)$ will allow us to evaluate the integral. However

$$\vec{r}\,' - \vec{w}(t_r) = \vec{r}\,' - \vec{w}(t - \tfrac{\mathscr{r}}{c}) = \vec{r}\,' - \vec{w}\left(t - \frac{|\vec{r} - \vec{r}\,'|}{c}\right)$$

contains $\vec{r}\,'$ in two locations and such a simple substitution will not be easy.

This requires careful manipulation of the Dirac delta function. We need to evaluate integrals of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{r}\,')\ \delta^3(\vec{r}\,'-\vec{r}_0)\, dx'dy'dz'$$

Then provided that $\vec{r}_0$ is independent of x', y', z', we get that

$$\int f(\vec{r}\,')\,\delta^3(\vec{r}\,'-\vec{r}_0)\,d\tau' = f(\vec{r}_0)$$

Strictly we are using

$$\delta^3(\vec{r}\,'-\vec{r}_0) = \delta^3\big((x'-x_0)\hat{x} + (y'-y_0)\hat{y} + (z'-z_0)\hat{z}\big)$$

$$= \delta(x'-x_0)\,\delta(y'-y_0)\,\delta(z'-z_0)$$

The issues with the three-dimensional delta function can be reduced to issues with the one-dimensional delta function. Then

$$\int_{-\infty}^{\infty} f(x')\,\delta(x'-x_0)\,dx' = f(x_0) \quad \text{ONLY IF } x_0 \text{ does not depend on } x'$$

We will see that this is not the case for retarded potentials

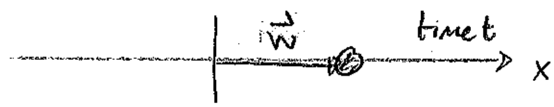
1 **Retarded Dirac delta function for a moving point charge.**

Consider a point charge particle moving along the $+x$-axis with speed v and which is at the origin at $t = 0$. Suppose that we aimed to determine the field produced by this at the origin at any time t.

a) Determine an expression for
$$\mathbf{r}' - \mathbf{w}(t_r)$$
in terms of x', y' and z'.

b) Express $\delta^3(\mathbf{r}' - \mathbf{w}(t_r))$ as a product of three one dimensional delta functions.

Answer: a)



$$\vec{w}(t) = vt\,\hat{x}$$

$$\Rightarrow \quad \vec{w}(t_r) = v(t - \mathscr{r}/c)\,\hat{x}$$

Then $\quad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = -\vec{r}\,' \quad \Rightarrow \quad \mathscr{r} = r'$ Thus

$$\vec{r}\,' - \vec{w}(t_r) = \vec{r}\,' - \left(vt - \frac{v}{c} r'\right)\hat{x}$$

Note that we need to keep the generic form for $\vec{r}\,'$ for the integral. So

$$\vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$

$$r' = \sqrt{x'^2 + y'^2 + z'^2}$$

$$\Rightarrow \quad \vec{r}\,' - \vec{w}(t_r) = \left[x' - vt + \frac{v}{c}\sqrt{x'^2 + y'^2 + z'^2}\right]\hat{x} + y'\hat{y} + z'\hat{z}$$

b) $$\delta^3(\vec{r}\,' - \vec{w}(t_r)) = \delta\left(x' - vt + \frac{v}{c}\sqrt{x'^2 + y'^2 + z'^2}\right)\delta(y')\,\delta(z')$$

If we were to integrate

$$\int_{\text{all space}} f(x', y', z')\,dx'\,dy'\,dz' = \int f(x', 0, 0)\,\delta\left(x' - vt + \frac{v}{c}x'\right)dx'$$

This brings up the issue of evaluating

$$\int_{-\infty}^{\infty} f(x')\ \delta(g(x'))\,dx'$$

where g is a function of one variable. To manage this we

① transform variables

$$u = g(x')$$

② use an inverse transformation

$$x' = g^{-1}(u)$$

③ transform the integral

$$\int_{-\infty}^{\infty} f(x')\ \delta(g(x'))\,dx' = \int_{-\infty}^{\infty} f(g^{-1}(u))\ \delta(u)\ \left(\frac{dx'}{du}\right) du$$

where

$$\frac{dx'}{du} = \frac{dg^{-1}}{du} \qquad \Rightarrow \int_{-\infty}^{\infty} f(g^{-1}(u))\ \delta(u)\ \frac{dg^{-1}}{du}\,du$$

④ evaluate:

$$\int_{-\infty}^{\infty} f(x')\ \delta(g(x'))\,dx' = f(g^{-1}(0))\left(\frac{dg^{-1}}{du}\right)\Big|_{u=0}$$

2 Retarded Dirac delta function integral evaluation

Consider a point charge particle moving along the $+x$-axis with speed v and which is at the origin at $t = 0$. Suppose that we aimed to determine the field produced by this at the origin at any time t. We aim to evaluate

$$\int f(x')\delta\left(x' - vt + \frac{v}{c}x'\right) \mathrm{d}x'.$$

a) Using

$$u = g(x') = x' - vt + \frac{v}{c}x'$$

determine the inverse function $g^{-1}(u)$.

b) Determine

$$\frac{dg^{-1}}{du}$$

and show that

$$\frac{dg^{-1}}{du} = 1/\frac{du}{dx'}$$

c) Evaluate

$$\int f(x')\delta\left(x' - vt + \frac{v}{c}x'\right) \mathrm{d}x'.$$

Answer: a) $x' = g^{-1}(u)$ But $u = x'(1+v/c) - vt \Rightarrow \frac{u+vt}{1+v/c} = x'$

$$\Rightarrow g^{-1}(u) = \frac{u+vt}{1+v/c}$$

b) $\frac{dg^{-1}}{du} = \frac{1}{1+v/c}$ and $\frac{du}{dx'} = 1 + v/c$ so $\frac{dg^{-1}}{du} = 1/\frac{du}{dx'}$

c) $\int f(x')\,\delta(x' - vt + \frac{v}{c}x')\,dx' = f(g^{-1}(0))\,\frac{dg^{-1}}{du}\Big|_{u=0}$

Now $g^{-1}(0) = \frac{vt}{1+v/c}$ and $\frac{dg^{-1}}{du} = \frac{1}{1+v/c}$

$$\Rightarrow \int f(x')\,\delta(x' - vt + \frac{v}{c}x')\,dx' = f\left(\frac{vt}{1+v/c}\right)\frac{1}{1+v/c}$$

In general, if $x' = g^{-1}(u)$ then

$$\frac{dg^{-1}}{du} = 1 \Big/ \frac{du}{dx'} = 1 \Big/ \frac{dg}{dx}$$

and thus

$$\boxed{\int_{-\infty}^{\infty} f(x')\, \delta(g(x'))\, dx' = f(g^{-1}(0)) \Big/ \left.\frac{dg}{dx'}\right|_{g^{-1}(0)}}$$

Thus evaluation this type of delta-function integral hinges on

* identifying $g(x')$
* finding the inverse to $g(x')$ and evaluating it at one point.

Three dimensional delta function evaluation

We will need to evaluate integrals of the form:

$$I = \int f(\vec{r}')\, \delta(\vec{r}' - \vec{w}(t_r))\, d\tau'$$

where $t_r = t - \frac{1}{c}|\vec{r} - \vec{r}'|$. The associated three dimensional integral is:

$$I = \iiint f(x',y',z')\, \delta(x' - w_x(t_r))\, \delta(y' - w_y(t_r))\, \delta(z' - w_z(t_r))\, dx'dy'dz'$$

So we define

$$\vec{u} = \vec{r}' - \vec{w}(t_r) = \vec{g}(\vec{r}')$$

where $\vec{g}$ is a vector function. Specifically

$$u_x = g_x(\vec{r}') = x' - w_x(t_r)$$

$$= x' - w_x\left(t - \frac{1}{c}\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}\right)$$

etc... Then there is an inverse function

$$\vec{r}' = \vec{g}^{-1}(\vec{u})$$

and

$$I = \int f(\vec{g}^{-1}(\vec{u}))\, \delta^3(\vec{u})\, dx'dy'dz'$$

We need to transform the volume element. The rule is

$$dx'dy'dz' = \frac{1}{J}\, du_x\, du_y\, du_z$$

where the Jacobian is

$$J = \begin{vmatrix} \frac{\partial u_x}{\partial x'} & \frac{\partial u_y}{\partial x'} & \frac{\partial u_z}{\partial x'} \\ \frac{\partial u_x}{\partial y'} & \frac{\partial u_y}{\partial y'} & \frac{\partial u_z}{\partial y'} \\ \frac{\partial u_x}{\partial z'} & \frac{\partial u_y}{\partial z'} & \frac{\partial u_z}{\partial z'} \end{vmatrix} \equiv J(\vec{g}^{-1}(\vec{u}))$$

Thus the integral is

$$I = \int f(\vec{g}^{-1}(\vec{u}))\, \delta^3(\vec{u})\, \frac{1}{J(\vec{g}^{-1}(\vec{u}))}\, d^3u$$

$$= \frac{f(\vec{g}^{-1}(0))}{J(\vec{g}^{-1}(0))}$$

Note that $\vec{g}^{-1}(0)$ is the value of $\vec{r}'$ such that $\vec{u}=0$.
Thus it is the value of $\vec{r}'$ such that

$$\vec{r}' = \vec{w}(t_r)$$

so we get

$$\int f(\vec{r}')\, \delta(\vec{r}' - \vec{w}(t_r))\, d\tau' = \frac{f}{J}\Bigg|_{\text{evaluated at } \vec{r}' \text{ that satisfies } \vec{r}' = \vec{w}(t_r)}$$

In this case the Jacobian is determine via:

$$\boxed{\begin{array}{l} \text{If} \quad \vec{u} = \vec{r}\,' - \vec{w}(t_r) \ \text{ then} \\ \\ J = 1 - \dfrac{1}{c}\dfrac{\vec{v}\cdot\vec{\mathscr{r}}}{\mathscr{r}} \end{array}}$$

Proof:

$$\frac{du_x}{dx'} = 1 - \frac{dw_x}{dx'} = 1 - \frac{dw_x}{dt}\frac{\partial t_r}{\partial x'} = 1 - V_x \frac{\partial t_r}{\partial x'}$$

$$\frac{du_y}{dy'} = 1 - V_y \frac{\partial t_r}{\partial y'}$$

$$\frac{\partial u_z}{\partial z'} = 1 - V_z \frac{\partial t_r}{\partial z'}$$

Then

$$\frac{\partial u_x}{\partial y'} = -\frac{dw_x}{dt}\frac{\partial t_r}{\partial y'}$$

$$\Rightarrow \quad J = \begin{vmatrix} 1 - V_x \frac{\partial t_r}{\partial x'} & -V_y \frac{\partial t_r}{\partial x'} & -V_z \frac{\partial t_r}{\partial x'} \\ -V_x \frac{\partial t_r}{\partial y'} & 1 - V_y \frac{\partial t_r}{\partial y'} & -V_z \frac{\partial t_r}{\partial y'} \\ -V_x \frac{\partial t_r}{\partial z'} & -V_y \frac{\partial t_r}{\partial z'} & 1 - V_z \frac{\partial t_r}{\partial z'} \end{vmatrix}$$

$$= \left(1 - V_x \frac{\partial t_r}{\partial x'}\right)\left[\left(1 - V_y \frac{\partial t_r}{\partial y'}\right)\left(1 - V_z \frac{\partial t_r}{\partial z'}\right) - V_y V_z \frac{\partial t_r}{\partial z'}\frac{\partial t_r}{\partial y'}\right]$$

$$- V_y \frac{\partial t_r}{\partial x'}\left[V_x V_z \frac{\partial t_r}{\partial z'}\frac{\partial t_r}{\partial y'} + \left(1 - V_z \frac{\partial t_r}{\partial z'}\right) V_x \frac{\partial t_r}{\partial y'}\right]$$

$$- V_z \frac{\partial t_r}{\partial x'}\left[V_x V_y \frac{\partial t_r}{\partial y'}\frac{\partial t_r}{\partial z'} + \left(1 - V_y \frac{\partial t_r}{\partial y'}\right) V_x \frac{\partial t_r}{\partial z'}\right]$$

$$\Rightarrow \quad J = \left(1 - v_x \frac{\partial t_r}{\partial x'}\right)\left(1 - v_y \frac{\partial t_r}{\partial y'} - v_z \frac{\partial t_r}{\partial z'}\right) - v_y \frac{\partial t_r}{\partial x'} v_x \frac{\partial t_r}{\partial y'}$$

$$- v_z \frac{\partial t_r}{\partial x'} v_x \frac{\partial t_r}{\partial z'}$$

$$= 1 - v_x \frac{\partial t_r}{\partial x'} - v_y \frac{\partial t_r}{\partial y'} - v_z \frac{\partial t_r}{\partial z'}$$

Now

$$t_r = t - \frac{1}{c}\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2}$$

$$\Rightarrow \quad \frac{\partial t_r}{\partial x'} = + \frac{1}{c} \frac{(x-x')}{\sqrt{}} = \frac{1}{c\,\mathscr{r}}(x-x')$$

$$\frac{\partial t_r}{\partial y'} = \frac{1}{c\,\mathscr{r}}(y-y')$$

$$\frac{\partial t_r}{\partial z'} = \frac{1}{c\,\mathscr{r}}(z-z')$$

gives:

$$J = 1 - \frac{1}{c\,\mathscr{r}} \vec{v} \cdot \vec{\mathscr{r}}$$

$$= 1 - \frac{1}{c} \frac{\vec{v} \cdot \vec{\mathscr{r}}}{\mathscr{r}} \qquad \blacksquare$$

This gives that

$$\boxed{\int f(\vec{r}\,')\, \delta(\vec{r}\,' - \vec{w}(t_r))\, d\tau' = f(\vec{r}\,')\left[1 - \frac{1}{c}\frac{\vec{v}\cdot\vec{\mathscr{r}}}{\mathscr{r}}\right]^{-1}\Bigg|_{\text{evaluated at } \vec{r}\,' \text{ s.t. } \vec{r}\,' = \vec{w}(t_r)}}$$

Retarded position

The evaluation of the retarded potential integrals requires a special value of $\vec{r}\,'$. This must satisfy

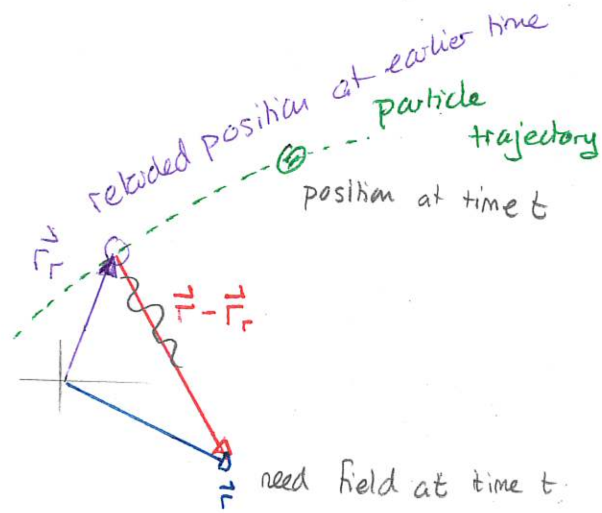
$$\vec{r}\,' = \vec{w}(t_r) = \vec{w}\left(t - \frac{1}{c}|\vec{r} - \vec{r}\,'|\right)$$

We call the value that satisfies this the retarded position $\vec{r}_r$. Thus

The retarded position satisfies

$$\vec{r}_r = \vec{w}\left(t - \frac{1}{c}|\vec{r} - \vec{r}_r|\right)$$

This has a physical interpretation as the location that the particle would have to have been at so that a signal propagating from that location would arrive at $\vec{r}$ at time t.



Note that

$$\vec{r}_r = \vec{w}(t_r)$$

$$t_r = t - \frac{1}{c}|\vec{r} - \vec{r}_r| \quad \Rightarrow \quad |\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

We can find this by

Identify field location $\vec{r}$ and time t $\longrightarrow$ Find retarded time t_r by solving

$$|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$$

Then

$$\vec{r}_r = \vec{w}(t_r)$$

Then we would get

$$\boxed{\int f(\vec{r}\,')\,\delta(\vec{r}\,' - \vec{w}(t_r))\,d\tau' = f(\vec{r}_r)\left[1 - \frac{1}{c}\frac{\vec{v}\cdot\vec{\mathscr{r}}_r}{\mathscr{r}_r}\right]^{-1}}$$

where

$$\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r$$

is the retarded separation vector

Using

$$f(\vec{r}\,') = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r} - \vec{r}\,'|}$$

would then yield the vector potential

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}_r}\frac{1}{1 - \frac{1}{c}\vec{v}\cdot\vec{\mathscr{r}}_r/\mathscr{r}_r}$$

3 Retardation for a particle traveling with constant velocity

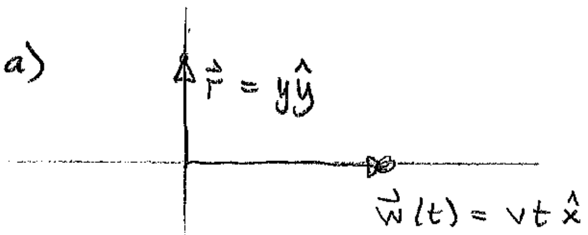
A charged particle travels with constant velocity along the $+x$ axis, passing the origin at $t = 0$. Suppose that one wants to determine the scalar potential at $\mathbf{r} = y\hat{\mathbf{y}}$ at time $t > 0$.

a) Determine an expression for the retarded time.

b) Suppose that $v = c/\sqrt{2}$. Determine an expression for the retarded position.

c) Suppose that $v = c/\sqrt{2}$. Determine an expression for the magnitude of the retarded separation vector.

Answer: a)



Solve: $|\vec{r} - \vec{w}(t_r)| = c(t - t_r)$

$$\Rightarrow |y\hat{y} - vt_r\hat{x}| = c(t - t_r)$$

$$\Rightarrow y^2 + v^2 t_r^2 = c^2(t - t_r)^2 = c^2 t_r^2 - 2c^2 t t_r + c^2 t^2$$

$$\Rightarrow t_r^2\left[c^2 - v^2\right] + t_r(-2c^2 t) + c^2 t^2 - y^2$$

$$\Rightarrow t_r = \frac{2c^2 t \pm \sqrt{4c^4 t^2 - 4(c^2t^2 - y^2)(c^2 - v^2)}}{2(c^2 - v^2)}$$

$$= \frac{c^2 t \pm \sqrt{c^4 t^2 - (c^2t^2 - y^2)(c^2 - v^2)}}{c^2 - v^2}$$

$$= \frac{c^2 t \pm \sqrt{\cancel{c^4t^2} - \cancel{c^4t^2} + c^2t^2v^2 + y^2c^2 - y^2v^2}}{c^2 - v^2}$$

$$= \frac{c^2 t \pm \sqrt{c^2v^2t^2 + y^2(c^2 - v^2)}}{c^2 - v^2}$$

Only the negative root gives $t_r < t$. Thus

$$t_r = \frac{1}{1 - v^2/c^2}\left[\, t - \sqrt{\frac{v^2t^2}{c^2} + \frac{y^2}{c^2}\left(1 - v^2/c^2\right)}\,\right]$$

b) here $\frac{v^2}{c^2} = \frac{1}{2}$ and $1 - v^2/c^2 = \frac{1}{2}$ gives:

$$t_r = 2\left[t - \sqrt{\frac{t^2}{2} + \frac{1}{2}\,y^2/c^2}\right]$$

$$= 2\left[t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\right]$$

then $\vec{r}_r = \vec{w}(t_r)$

$$= v t_r \hat{x} = 2v\left[t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\right]\hat{x}$$

c) $\vec{\mathscr{r}}_r = \vec{r} - \vec{r}_r = y\hat{y} - 2v\left[t - \frac{1}{\sqrt{2}}\sqrt{t^2 + y^2/c^2}\right]\hat{x}$