Phys 312 2025

Lecture 19

Thurs: 10.3.1
Fri: HW by 5pm

Potentials in electromagnetic theory

Electromagnetism can be formulated in terms of two potentials

* scalar potential $V(\vec{r},t)$
* vector " $\vec{A}(\vec{r},t)$.

Fields are computed from these by

$$\boxed{\begin{aligned}\vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}}$$

In general the potentials are determined from source current and charge densities by solving:

$$\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$$

There are many possible potentials that give the same fields; this is the gauge choice. It is always possible to choose a potential in the Lorentz gauge. Here

The potential is in the Lorentz gauge iff

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$$

and then:

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

This gives a set of four uncoupled equations:

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2 A_x - \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = -\mu_0 J_x$$

$$\vdots$$

These all have the form

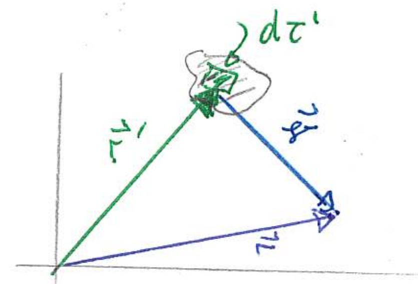
$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = g$$

Can we provide a generic solution for these? Consider the comparable expression for electrostatic potential $V(\vec{r},t)$. Then

$$\nabla^2 V = -\rho/\epsilon_0$$

has solution

$$V = \frac{1}{4\pi}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

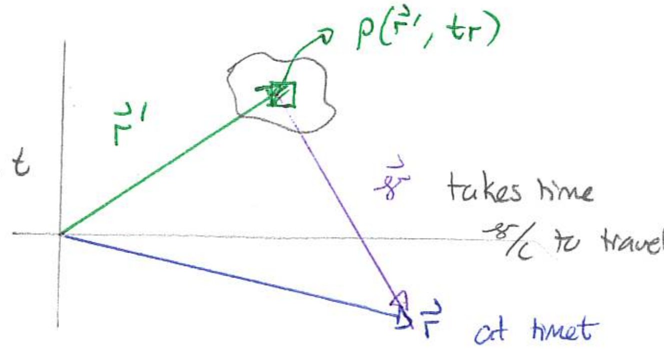



where $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$.

We seek a comparable expression for $f$ in terms of $g$. We might expect, that on physical grounds a change in $\rho(\vec{r}\,',t)$ over time will take time to propagate to the field point $\vec{r}$. It would appear that the signal might take time $\mathscr{r}/c$ to travel and thus the potential at $\vec{r}$ at time $t$ is determined by $\rho(\vec{r},t_r)$ where

$$t_r = t - \mathscr{r}/c$$

is an earlier time, retarded by amount $\mathscr{r}/c$.




This is true.

In general

A solution to

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r}, t)$$

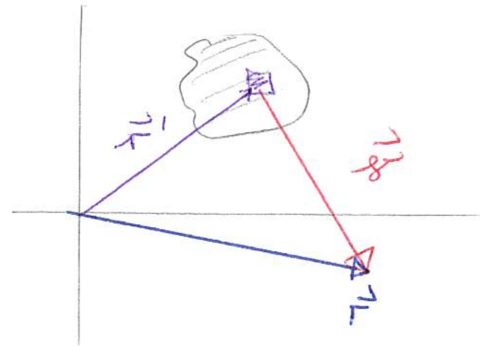
is

$$f(\vec{r}, t) = -\frac{1}{4\pi}\int_{\text{all space}} \frac{g(\vec{r}', t_r)}{\mathscr{r}}\, d\tau'$$

where $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

$t_r = t - \mathscr{r}/c$

is the retarded time




Proof. The proof involves computing $\nabla^2 f$ and $\frac{\partial^2 f}{\partial t^2}$ from the solution form:

First

$$\frac{\partial f}{\partial t} = -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}}\frac{\partial}{\partial t} g(\vec{r}, t_r)\, d\tau'$$

$$= -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}}\frac{\partial g}{\partial t}\frac{\partial t_r}{\partial t}\, d\tau'$$

Then $\frac{\partial t_r}{\partial t} = \frac{\partial}{\partial t}\left(t - \mathscr{r}/c\right) = 1$. gives

$$\frac{\partial f}{\partial t} = -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}}\frac{\partial g}{\partial t}\, d\tau'$$

Now

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}}\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial t}\right) d\tau' = -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}}\frac{\partial^2 g}{\partial t^2}\frac{\partial t_r}{\partial t}\, d\tau'$$

Thus:

$$\boxed{\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4\pi}\int \frac{1}{\mathscr{r}} \frac{\partial^2 g}{\partial t^2}\, d\tau'} \qquad -(A)$$

Now consider

$$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f$$

↳ unprimed.

Then

$$\vec{\nabla} f = -\frac{1}{4\pi}\int \left[\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) g + \frac{1}{\mathscr{r}}\vec{\nabla} g\right] d\tau'$$

and

$$\nabla^2 f = -\frac{1}{4\pi}\int \left[\nabla^2\left(\frac{1}{\mathscr{r}}\right) g + 2\,\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)\cdot\vec{\nabla} g + \frac{1}{\mathscr{r}}\nabla^2 g\right] d\tau' \qquad -(B)$$

Standard differentiation gives

$$\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\, \delta^3(\vec{\mathscr{r}})$$

$$\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Then

$$\vec{\nabla} g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z}$$

$$= \frac{\partial g}{\partial t}\left[\frac{\partial t_r}{\partial x}\hat{x} + \frac{\partial t_r}{\partial y}\hat{y} + \frac{\partial t_r}{\partial z}\hat{z}\right] = \frac{\partial g}{\partial t}\vec{\nabla}(t_r)$$

$$\nabla^2 g = \frac{\partial^2 g}{\partial t^2}(\vec{\nabla} t_r)\cdot(\vec{\nabla} t_r) + \frac{\partial g}{\partial t}\nabla^2(t_r)$$

Thus (B) gives:

$$\nabla^2 f = \int g(\vec{r}', t_r)\, \delta^3(\vec{\mathscr{r}})\, d\tau' - \frac{1}{2\pi}\int \left[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\vec{\nabla} g\, d\tau' + \frac{1}{2\mathscr{r}}\nabla^2 g\right] d\tau'$$

The first requires evaluation at $\vec{\mathscr{r}} = 0$

$$\Rightarrow \vec{r}\,' = \vec{r}$$

$$\Rightarrow t_r = t.$$

Thus

$$\nabla^2 f = g(\vec{r},t) - \frac{1}{2\pi}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\frac{\partial g}{\partial t}\,\vec{\nabla}(t_r)\,d\tau' - \frac{1}{4\pi}\int\frac{1}{\mathscr{r}}\left[\frac{\partial g}{\partial t}\nabla^2(t_r) + \frac{\partial^2 g}{\partial t^2}(\vec{\nabla}t_r)\cdot(\vec{\nabla}t_r)\right]d\tau'$$

Now $\vec{\nabla} t_r = \cancel{\vec{\nabla} t} - \frac{1}{c}\vec{\nabla}\mathscr{r} = -\frac{1}{c}\hat{\mathscr{r}}$

$$\nabla^2 t_r = -\frac{1}{c}\vec{\nabla}\cdot(\hat{\mathscr{r}}) = +\frac{1}{c}\frac{2}{\mathscr{r}}$$

Thus

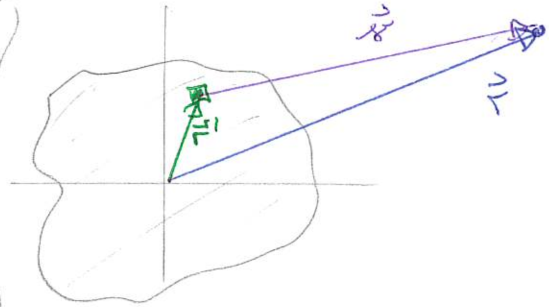
$$\nabla^2 f = g(\vec{r},t) + \frac{1}{2\pi}\left(+\frac{1}{c}\right)\int \cancel{\frac{1}{\mathscr{r}^2}\frac{\partial g}{\partial t}\,d\tau'} + \frac{1}{4\pi}\int\frac{1}{\mathscr{r}}\left[\cancel{\frac{\partial g}{\partial t}\left(+\frac{1}{c}\frac{2}{\mathscr{r}}\right)} + \frac{\partial^2 g}{\partial t^2}\frac{1}{c^2}\right]d\tau'$$

$$\nabla^2 f = g(\vec{r},t) + \frac{1}{c^2}\frac{1}{4\pi}\int\frac{1}{\mathscr{r}}\frac{\partial^2 g}{\partial t^2}d\tau'$$

$$= g(\vec{r},t) + \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \qquad \text{from (A)}$$

Thus $\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = g(\vec{r},t)$

When applied to the potentials this gives the <u>retarded potentials</u>




want potentials here at time $t$

region with time varying charge and current densities

$\rho(\vec{r}',t)$

$\vec{J}(\vec{r}',t)$

For every $\vec{r}'$ form

$\vec{\mathscr{r}} = \vec{r} - \vec{r}'$

and determine

$\mathscr{r} = |\vec{r} - \vec{r}'|$

also form

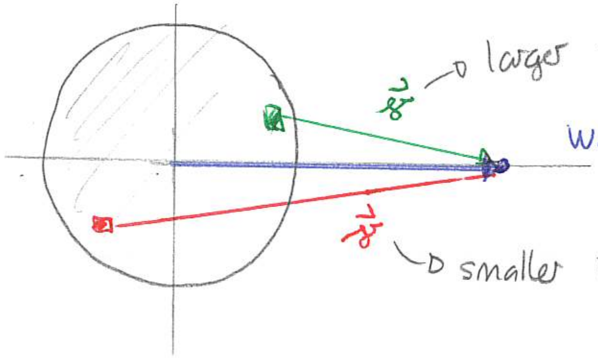
$t_r = t - \mathscr{r}/c$

The potentials are

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\mathscr{r}}\,d\tau'$$

Note that the integrals are complicated by the fact that $t_r$ varies throughout the charge distribution.




larger $t_r$ for this (evaluated at a later time)

want $V$ here at time $t$

smaller $t_r$ for this (evaluated at an earlier time)

## 1 Time-varying spherical potential

A sphere with radius $R$ has a charge density

$$\rho(\mathrm{r}', t) = \frac{Q}{2\pi R^2 r'} e^{-t/T}$$

where $T > 0$ is a constant with units of time and $Q$ is the total charge within the sphere at time $t = 0$. Consider the potentials as the center of the sphere at time $t$.

a) Determine expressions for the retarded times for contributions from the center of the sphere and the edge of the sphere.

b) Determine an expression for the scalar potential at the center of the sphere at any time $t$.

c) Describe how you might determine the current density associated with this charge density and how you might use it to determine the vector potential at the center of the sphere.

Answer: a) At the center $\vec{r} = 0$ and $\vec{\mathscr{r}} = \vec{r} - \vec{r}' = -r'$

$\Rightarrow \mathscr{r} = r'$

Then $t_r = t - \mathscr{r}/c \Rightarrow t_r = t - r'/c$

Contribution from center ($r' = 0$) $\quad t_r = t \quad$ (no delay)

" " edge ($r' = R$) $\quad t_r = t - R/c$ (earlier).

b)

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}', t_r)}{\mathscr{r}} d\tau' \qquad \left.\begin{array}{l} 0 \le r' \le R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\} d\tau' = r'^2 dr' d\theta' d\phi'$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R^2} \int_0^R dr' r'^2 \underbrace{\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta'}_{4\pi} \left(e^{-(t - r'/c)\frac{1}{T}} \frac{1}{r'}\right) \frac{1}{r'}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R^2} e^{-t/T} \, 4\pi \int_0^R dr' \, e^{+r'/cT}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q}{2R^2} e^{-t/T} \, cT \, e^{r'/cT} \Big|_0^R$$

$$\Rightarrow \quad V(\vec{r},t) = \frac{Q}{4\pi\epsilon_0}\frac{c\tau}{2R^2} e^{-t/\tau}\left[e^{R/c\tau} - 1\right]$$

c) One could use the continuity equation (unprimed co-ordinate).

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\frac{Q}{4\pi R^2}\frac{1}{r}\left(-\frac{1}{\tau}\right)e^{-t/\tau} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = \frac{Q}{4\pi R^2}\frac{1}{\tau}e^{-t/\tau}\frac{1}{r}$$

Assuming $\vec{J}$ is symmetrical.

$$\vec{\nabla}\cdot\vec{J} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 J_r\right)$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2 J_r\right) = \frac{Q}{4\pi R^2}\frac{1}{\tau}e^{-t/\tau}\frac{1}{r}$$

$$\Rightarrow \frac{d}{dr}\left(r^2 J_r\right) = \frac{Q}{4\pi R^2}\frac{1}{\tau}e^{-t/\tau}\,r$$

$$\Rightarrow r^2 J_r = \frac{Q}{4\pi R^2}\frac{1}{\tau}e^{-t/\tau}\left[\frac{r^2}{2} + \text{const}\right].$$

$$\Rightarrow J_r = \frac{Q}{4\pi R^2}\frac{1}{\tau}e^{-t/\tau}\left[\frac{1}{2} + \frac{\text{const}}{r^2}\right]$$

One could then use the retarded potential to determine $\vec{A}$.

## Point source charges

We eventually want to determine the fields produced by point source charges. To do this requires a charge density which

* will be zero at all locations except that of the point charge

* gives a total charge $q$ when integrated over all space.

The model for this will use the three dimensional Dirac delta function $\delta^3(\vec{r}-\vec{r}_0)$. This satisfies:

1) $\delta^3(\vec{r}-\vec{r}_0) = 0$ if $\vec{r} \neq \vec{r}_0$

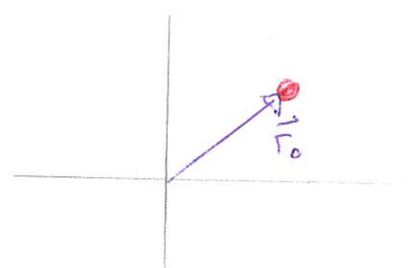
2) $$\int_{\text{all space}} f(\vec{r})\, \delta^3(\vec{r}-\vec{r}_0)\, d\tau = f(\vec{r}_0)$$

Additionally one can prove

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\, \delta^3(\vec{r})$$

We can then consider a stationary point charge whose magnitude varies with time. Then

$$\vec{\rho}(\vec{r},t) = q(t)\, \delta^3(\vec{r}-\vec{r}_0)$$




We can shift the origin so that $\vec{r}_0 = 0$. Then

$$\rho(\vec{r},t) = q(t)\, \delta^3(\vec{r})$$

Then the continuity equation requires a current $\vec{J}$ via

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \dot{q}(t)\, \delta^3(\vec{r}) = -\vec{\nabla}\cdot\vec{J} \Rightarrow \vec{J} = -\frac{\dot{q}(t)}{4\pi}\frac{\hat{r}}{r^2}$$

## 2 Potential from a stationary point charge

Consider a single time varying point charge at the origin. The charge density is

$$\rho(\mathbf{r}', t) = q(t)\delta^3(\mathbf{r}')$$

and the associated current density is

$$\mathbf{J}(\mathbf{r}', t) = -\frac{\dot{q}(t)}{4\pi r'^2}\,\hat{\mathbf{r}}'$$

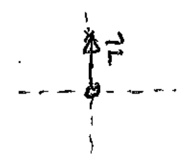
where $\dot{q}(t)$ is the time derivative of $q(t)$. These are readily shown to satisfy the continuity equation.

a) Determine the retarded potential $V(\mathbf{r}, t)$.

b) Using a symmetry argument, show that the retarded potential $\mathbf{A}(\mathbf{r}, t)$ only has a radial component and that this is independent of angle. Use this result to determine an expression for the magnetic field.

c) Use the magnetic field and one of Maxwell's equations to determine the electric field.

Answer: a)




$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{\mathscr{r}}\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{q(t_r)\,\delta^3(\vec{r}')}{\mathscr{r}}\,d\tau'$$

Now $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ and $t_r = t - \mathscr{r}/c = t - \frac{|\vec{r}-\vec{r}'|}{c}$. Thus

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int q\left[t - \frac{|\vec{r}-\vec{r}'|}{c}\right]\frac{1}{|\vec{r}-\vec{r}'|}\,\delta^3(\vec{r}')\,d\tau'$$

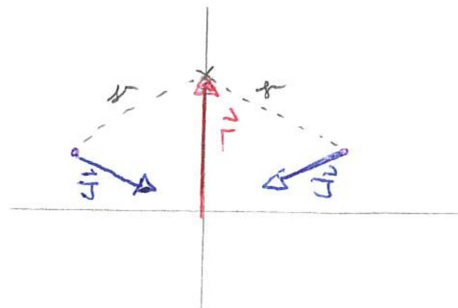
$$= \frac{1}{4\pi\epsilon_0}\, q\left(t - \frac{r}{c}\right)\frac{1}{r} \Rightarrow V(\vec{r},t) = \frac{1}{4\pi\epsilon_0 r}\, q(t - r/c)$$

b) $$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{\mathscr{r}}\,d\tau'$$

This requires adding $\vec{J}(\vec{r}', t_r)$ over all locations.

Consider a field location along $\hat{z}$

Then consider the contributions from the two mirror images. The values of $\mathscr{r}$ are the same for these two. So are $t_r$. Then we would add two $\vec{J}$ vectors with the same magnitude. Clearly the $\hat{x}$ and $\hat{y}$ components cancel. Only the $\hat{z}$ component survives. Thus we expect




$$\vec{A}(\vec{r},t) = A_r(r,t)\,\hat{r}$$

In detail:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(-\frac{1}{4\pi}\right)\int \frac{\dot{q}(t-\mathscr{r}/c)}{r'^2}\frac{\hat{r}'}{\mathscr{r}}\,d\tau'.$$

Now $\mathscr{r} = |\vec{\mathscr{r}}\cdot\vec{\mathscr{r}}|^{1/2} = \sqrt{(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')}$

$$= (r^2+r'^2-2\vec{r}\cdot\vec{r}')^{1/2}$$

Then $\vec{r} = z\hat{z}$ gives

$$\mathscr{r} = (z^2+r'^2-2z\,\hat{z}\cdot\vec{r}')^{1/2}$$

$$= (z^2+r'^2-2zr'\cos\theta')^{1/2}$$

Thus

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{(4\pi)^2}\int \frac{\dot{q}\left(t-\sqrt{z^2+r'^2-2zr'\cos\theta'}/c\right)}{r'^2\sqrt{z^2+r'^2-2zr'\cos\theta'}}\,\hat{r}'\,d\tau'$$

But $d\tau' = r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$ gives

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{(4\pi)^2}\int_0^\infty dr'\,\frac{r'^2}{r'^2}\int_0^{2\pi} d\phi'\int_0^{\pi} d\theta'\sin\theta'\;\frac{\dot{q}\left(t-\sqrt{z^2+r'^2-2zr'\cos\theta'}/c\right)}{\sqrt{z^2+r'^2-2zr'\cos\theta'}}\,\hat{r}'$$

Note that

$$\hat{r}' = \cos\phi'\sin\theta'\,\hat{x} + \sin\phi'\sin\theta'\,\hat{y} + \cos\theta'\,\hat{z}$$

varies with the integration variables. The $\sin\phi'$ and $\cos\phi'$ terms integrate to zero

Thus.

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{(4\pi)^2}\int_0^\infty dr' \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \int_0^\pi d\theta' \sin\theta' \,\frac{\dot{q}\left(t - \sqrt{\phantom{z^2+r'^2-2zr'\cos\theta'}}/c\right)}{\sqrt{z^2+r'^2-2zr'\cos\theta'}}\cos\theta'\,\hat{z}$$

$$= -\frac{\mu_0}{8\pi}\int_0^\infty dr' \int_0^\pi d\theta' \sin\theta'\cos\theta' \,\frac{\dot{q}\left(t - \sqrt{z^2+r'^2-2zr'\cos\theta'}/c\right)}{\sqrt{z^2+r'^2-2zr'\cos\theta'}}\,\hat{z}$$

Although we cannot evaluate this exactly we can see that $\vec{A}(\vec{r})$ points radially and only depends on $r$. Thus

$$\vec{A} = A_r(r)\,\hat{r}$$

In general

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta \cancel{A_\phi}) - \cancel{\frac{\partial A_\theta}{\partial\phi}}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}} - \frac{\partial}{\partial r}(r\cancel{A_\phi})\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r\cancel{A_\theta}) - \cancel{\frac{\partial A_r}{\partial\theta}}\right]\hat{\phi}$$

$$\Rightarrow \vec{B} = 0$$

c) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{E} = 0$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = 0 \Rightarrow \frac{\partial\vec{E}}{\partial t} = -\frac{1}{\epsilon_0}\vec{J}$$

$$\Rightarrow \frac{\partial\vec{E}}{\partial t} = -\frac{1}{\epsilon_0}\left(\frac{-\dot{q}(t)}{4\pi r^2}\hat{r}\right)$$

$$\Rightarrow \frac{\partial\vec{E}}{\partial t} = \frac{\dot{q}(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

Thus

$$\vec{E} = \frac{q(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

Note that we could eventually determine $\vec{A}$ via

$$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$$

$$\Rightarrow \quad \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V - \vec{E}$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{q(t-r/c)}{r}\right) - \frac{q(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$= -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{1}{r}\right)q(t-r/c) - \frac{1}{4\pi\epsilon_0}\frac{1}{r}\vec{\nabla}\left[q(t-r/c)\right] - \frac{q(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}q(t-r/c) - \frac{1}{4\pi\epsilon_0}\frac{1}{r}\dot{q}(t-r/c)\left(-\frac{1}{c}\right)\underbrace{\vec{\nabla}r}_{\hat{r}} - \frac{q(t)}{4\pi\epsilon_0 r^2}\hat{r}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\left[q(t-r/c) - q(t)\right]\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{1}{rc}\dot{q}(t-r/c)\hat{r}$$