Thurs: Class exam 2

* Covers Ch 9
  Lectures 10 – 17
* Bring ~~previous~~ ½ single side letter shee
  - include Poynting vector etc...
* Previous exams
  2016 Exam 2 Q1, Q2a
  2021 Exam 2 Q1, Q2, Q3, ~~Q4~~.

Potential formulation of electromagnetic theory

Electromagnetic theory can be constructed in terms of potentials.

> There exist a scalar potential $V(\vec{r},t)$ and a vector potential $\vec{A}(\vec{r},t)$ such that fields are determined by
>
> $$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$
>
> $$\vec{B} = \vec{\nabla} \times \vec{A}$$

The potentials can be obtained from the source charges and currents by

$$\boxed{\begin{aligned} &\nabla^2 V + \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0 \\ &\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\left[\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right] = -\mu_0 \vec{J} \end{aligned}}$$

Such potentials automatically satisfy Maxwell's equations and the continuity equation.

Gauge transformations

The important physical quantities – electric and magnetic fields – are obtained from potentials by differentiation. Thus many different potentials can give the same fields.

For example, in electrostatics, $\frac{\partial \vec{A}}{\partial t} = 0$ and

$$\vec{E} = -\vec{\nabla} V \qquad \text{and} \qquad \nabla^2 V = -\rho/\epsilon_0$$

Thus if $V_1(\vec{r})$ is a solution to the Poisson equation then so is $V_2(\vec{r}) = V_1(\vec{r}) + V_0$ where V_0 is an overall constant. Note that $V_1(\vec{r})$ and $V_2(\vec{r})$ give the same fields.

We now explore this in general cases. The scheme is:

One pair of potentials V_1 and $\vec{A}_1$ $\longrightarrow$ Produce fields $\vec{E} = -\vec{\nabla} V_1 - \frac{\partial \vec{A}_1}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}_1$

Legitimate gauge transformation $\longleftarrow$ Fields are same

Another pair of potentials V_2 and $\vec{A}_2$ $\longrightarrow$ Produce exactly the same fields $\vec{E} = -\vec{\nabla} V_2 - \frac{\partial \vec{A}_2}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}_2$

What are the relationships such that this is valid?

$$\vec{E} \text{ equal} \Rightarrow -\vec{\nabla} V_1 - \frac{\partial \vec{A}_1}{\partial t} = -\vec{\nabla} V_2 - \frac{\partial \vec{A}_2}{\partial t} \Leftrightarrow \vec{\nabla}(V_2 - V_1) + \frac{\partial}{\partial t}(\vec{A}_2 - \vec{A}_1) = 0$$

$$\vec{B} \text{ equal} \Rightarrow \vec{\nabla} \times \vec{A}_1 = \vec{\nabla} \times \vec{A}_2 \Leftrightarrow \vec{\nabla} \times (\vec{A}_2 - \vec{A}_1) = 0$$

Consider the latter. This will be true if there is a scalar function $\lambda(\vec{r},t)$ such that $\vec{A}_2 - \vec{A}_1 = \vec{\nabla}\lambda$

$$\Rightarrow \quad \vec{A}_2 = \vec{A}_1 + \vec{\nabla}\lambda$$

Then the electric field equality is satisfied if

$$\vec{\nabla}(V_2 - V_1) + \frac{\partial}{\partial t}\vec{\nabla}\lambda = 0 \quad \Rightarrow \quad \vec{\nabla}\left[V_2 - V_1 + \frac{\partial \lambda}{\partial t}\right] = 0$$

A necessary and sufficient requirement for this to be true is

$$V_2 - V_1 + \frac{\partial \lambda}{\partial t} = V_0 \equiv \text{constant}.$$

We can set $V_0 = 0$ without any loss of generality. Thus

> If the potentials $V_2, \vec{A}_2$ are related to $V_1, \vec{A}_1$ by
>
> $$V_2 = V_1 - \frac{\partial \lambda}{\partial t}$$
>
> $$\vec{A}_2 = \vec{A}_1 + \vec{\nabla}\lambda$$
>
> where $\lambda = \lambda(\vec{r},t)$ is a scalar function, then these generate the same $\vec{E}, \vec{B}$ fields as $V_1, \vec{A}_1$.

The transformation

$$V_1 \to V_2 = V_1 - \frac{\partial \lambda}{\partial t}$$

$$\vec{A}_1 \to \vec{A}_2 = \vec{A}_1 + \vec{\nabla}\lambda$$

is called a gauge transformation. The choice of a particular pair of potentials is called a gauge choice.

Coulomb gauge

Suppose that the potentials satisfy $\vec{\nabla}\cdot\vec{A}=0$. Then these are obtained from source charges and currents by

$$\nabla^2 V = -\rho/\epsilon_0 \qquad \text{(Poisson's equation)}$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0 \frac{\partial^2\vec{A}}{\partial t^2} - \mu_0\epsilon_0 \frac{\partial}{\partial t}\vec{\nabla}V = -\mu_0\vec{J}$$

$$\vec{\nabla}\cdot\vec{A}=0$$

We could use techniques for solving Poisson's equation to get V. We could then substitute into the remaining equation to get $\vec{A}$. In electro and magnetostatic situations, $\vec{A}$ and V are true independent. Then

$$\nabla^2\vec{A} = -\mu_0\vec{J}$$

and this will admit the solution

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{J(\vec{r}\,')}{\mathscr{r}}\,d\tau'.$$

Suppose that $\vec{A}_1$ does not satisfy this. Can we find λ s.t. $\vec{A}_2=\vec{A}_1+\vec{\nabla}\lambda$ satisfies $\vec{\nabla}\cdot\vec{A}_2=0$. If so then $0=\vec{\nabla}\cdot\vec{A}_1+\nabla^2\lambda$ and we need only find λ s.t. $\nabla^2\lambda=-\vec{\nabla}\cdot\vec{A}_1$. This is Poisson's equation and it can usually be solved for λ. Thus we can always transform the potentials into the

Coulomb gauge:

$$\boxed{\begin{aligned}&\vec{\nabla}\cdot\vec{A}=0\\ \text{and}\quad &\nabla^2 V=-\rho/\epsilon_0\\ &\nabla^2\vec{A}-\mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2}-\mu_0\epsilon_0\frac{\partial}{\partial t}\vec{\nabla}V=-\mu_0\vec{J}\end{aligned}}$$

1 Coulomb gauge

Suppose that

$$V = \frac{x^2 B_0}{2}$$

and

$$\mathbf{A} = -xtB_0\hat{\mathbf{x}} + A_0 \sin{(kx - \omega t)}\hat{\mathbf{y}}$$

where A_0 and B_0 are constants.

a) Determine the electric and magnetic fields associated with these potentials.

b) Is $\mathbf{A}$ in the Coulomb gauge?

c) Determine a gauge transformation that transforms into the Coulomb gauge. Determine expressions for the potentials in this gauge.

d) Determine the fields using the Coulomb gauge potentials.

Answer: a) $\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$

$= -xB_0\hat{x} + xB_0\hat{x} + A_0\omega\cos(kx-\omega t)\hat{y} \Rightarrow \vec{E} = \omega A_0\cos(kx-\omega t)\hat{y}$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xtB_0 & A_0\sin(kx-\omega t) & 0 \end{vmatrix}$$

$= kA_0\cos(kx-\omega t)\hat{z} \Rightarrow \vec{B} = kA_0\cos(kx-\omega t)\hat{z}$

This is a plane electromagnetic wave traveling along the x-axis.

b) $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} = -tB_0 \neq 0$.

It's not in the Coulomb gauge.

c). We need to find $\lambda(\vec{r},t)$ so that

$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$

satisfies $\vec{\nabla}\cdot\vec{A}' = 0$

Then:

$$\vec{\nabla}\cdot\vec{A}' = 0 \Rightarrow \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda = 0$$

$$\Rightarrow -tB_0 + \nabla^2\lambda = 0$$

$$\Rightarrow \nabla^2\lambda = tB_0$$

There are many ways to satisfy

$$\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2} + \frac{\partial^2\lambda}{\partial z^2} = tB_0$$

We choose that which simplifies $V' = V - \frac{\partial\lambda}{\partial t}$ the most.

$\Rightarrow V' = \frac{x^2}{2}B_0 - \frac{\partial\lambda}{\partial t}$ is simplest.

The simplest would be $V' = 0 \Rightarrow \lambda = \frac{1}{2}tx^2B_0$. Then $\nabla^2\lambda = tB_0$ will ensure $\vec{\nabla}\cdot\vec{A}' = 0$. So

$$\lambda = \frac{1}{2}tx^2B_0 \Rightarrow V' = 0$$

$$\Rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$\Rightarrow \vec{A}' = -xtB_0\hat{x} + A_0\sin(kx-\omega t)\hat{y} + xtB_0\hat{x}$$

$$\Rightarrow \boxed{\begin{aligned}\vec{A}' &= A_0\sin(kx-\omega t)\hat{y}\\ V' &= 0\end{aligned}}$$

d) $\vec{E} = -\cancel{\vec{\nabla}V'} - \frac{\partial\vec{A}'}{\partial t} = \omega A_0\cos(kx-\omega t)\hat{y} \Rightarrow \vec{E} = \omega A_0\cos(kx-\omega t)\hat{y}$

$$\vec{B} = \vec{\nabla}\times\vec{A}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & A_0\sin(kx-\omega t) & 0 \end{vmatrix} \Rightarrow \vec{B} = kA_0\cos(kx-\omega t)\hat{z}$$

same as before

Lorentz gauge

We seek to decouple the equations for V and $\vec{A}$ by an appropriate gauge choice. If the potentials could satisfy

$$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}$$

then we would get

$$\nabla^2 V + \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = -\rho/\epsilon_0 \Rightarrow \nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$

$$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\left[\,..0\,\right] = -\mu_0\vec{J} \Rightarrow \nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Such potentials are in the Lorentz gauge.

> Thus if the potentials satisfy
>
> $$\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$$
>
> then they are in the Lorentz gauge. These potentials are related to source charges and currents by
>
> $$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$$
>
> $$\nabla^2\vec{A} - \mu_0\epsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

What if the potentials do not satisfy $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = 0$. We can transform

$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$V \to V' = V - \frac{\partial\lambda}{\partial t}$$

Then

$$\vec{\nabla}\cdot\vec{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2}$$

$$\Rightarrow \nabla^2\lambda - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \mu_0\epsilon_0\frac{\partial V}{\partial t}$$

We evaluate the r.h.s. using the original potentials. We then solve the second order differential equation for λ

$$\nabla^2\lambda - \mu_0\epsilon_0 \frac{\partial^2\lambda}{\partial t^2} = \underbrace{-\left[\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right]}_{\text{given}}$$

for λ. Such a solution exists fairly generally. So we can always find potentials in the Lorentz gauge. So now the strategy is

given charge density $\rho(\vec{r},t)$ →

Scalar potential satisfies

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

Only requires charge density

given current density $\vec{J}(\vec{r},t)$ →

Vector potential satisfies

$$\nabla^2\vec{A} - \mu_0\epsilon_0 \frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Only requires current density

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla} V - \frac{\partial\vec{A}}{\partial t}$$

2 Lorentz gauge

Suppose that

$$V = 0$$

and

$$\mathbf{A} = A_0 \sin(kx - \omega t)\hat{\mathbf{y}}$$

where A_0 and B_0 are constants. Are these potentials in the Lorentz gauge?

a) Are these potentials in the Lorentz gauge?

b) Determine source charge and current densities that might produce these potentials.

Answer: a) Check $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t} = \cancel{\frac{\partial A_y}{\partial y}} + \mu_0\epsilon_0 \cancel{\frac{\partial V}{\partial t}}^{0}$

$$= 0$$

Yes. They are in the Lorentz gauge.

b) $\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\rho/\epsilon_0$

$\Rightarrow \quad 0 = -\rho/\epsilon_0 \Rightarrow \rho = 0$

c) $\nabla^2 \vec{A} - \mu_0\epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$

$\Rightarrow \quad \nabla^2 A_0 \sin(kx-\omega t)\,\hat{y} - \mu_0\epsilon_0 (-\omega^2) A_0 \sin(kx-\omega t)\,\hat{y} = -\mu_0\vec{J}$

$\Rightarrow \quad -k^2 A_0 \sin(kx-\omega t)\,\hat{y} + \omega^2\mu_0\epsilon_0 A_0 \sin(kx-\omega t)\,\hat{y} = -\mu_0\vec{J}$

$\Rightarrow \quad A_0\left(\frac{\omega^2}{c^2} - k^2\right) \sin(kx-\omega t)\,\hat{y} = -\mu_0\vec{J}$

If $k = \omega/c$ then $\vec{J} = 0$.

So there are no continuous current and charge densities that produce such potentials. However, there may be a delta-function type of distribution that does this.