Phys 312 2025

Lecture 16

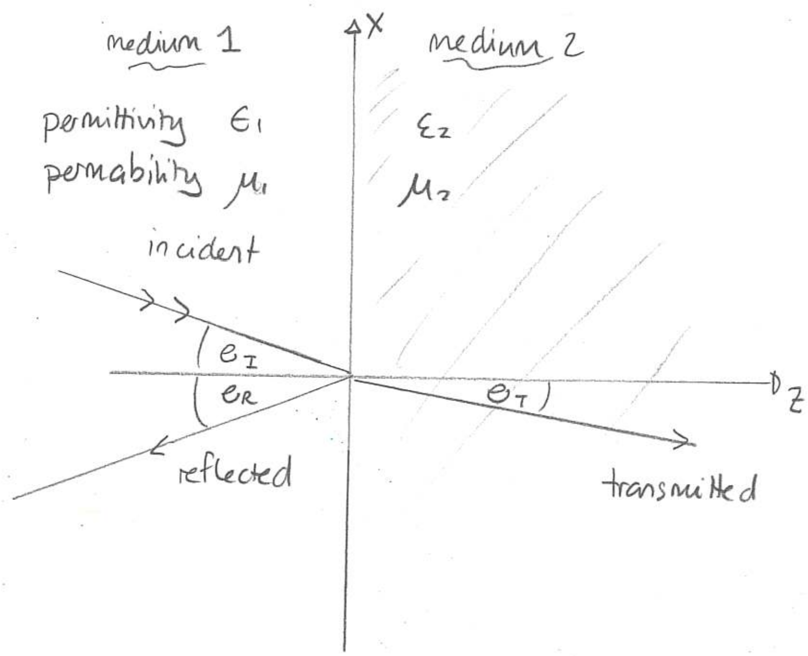
Tues: HW by 5pm

Thurs: Read 10.1.1, 10.1.2

~~**Fri:** HW by 5pm~~

Reflection and transmission at a surface

Recall that we are considering waves incident on a surface between two dielectric media. We were able to determine basic laws of geometric optics by considering sinusoidal plane waves.



1) incident

$$\vec{E}_I = \vec{E}_{0I}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)}$$

$$\vec{B}_I = \frac{1}{\omega_I}\left(\vec{k}_I \times \vec{E}_I\right)$$

2) reflected

$$\vec{E}_R = \vec{E}_{0R}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$$

$$\vec{B}_R = \frac{1}{\omega_R}\left(\vec{k}_R \times \vec{E}_R\right)$$

3) transmitted

$$\vec{E}_T = \vec{E}_{0T}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)} \qquad \vec{B}_T = \frac{1}{\omega_T}\left(\vec{k}_T \times \vec{E}_T\right)$$

Applying any of the boundary conditions at $z=0$ (for all points in that plane) gives the basic rules of geometric optics

These are:

1) the frequencies of all three waves are the same

$$\omega_I = \omega_R = \omega_T = \omega$$

2) the wavenumbers all lie in the same plane. This plane contains the normal to the surface and also the wavenumber vector of the incident wave.

3) the components of the three wavenumbers parallel to the interface are the same.

$$\Rightarrow \quad \theta_R = \theta_I$$

$$\text{AND} \quad n_1 \sin\theta_I = n_2 \sin\theta_T$$

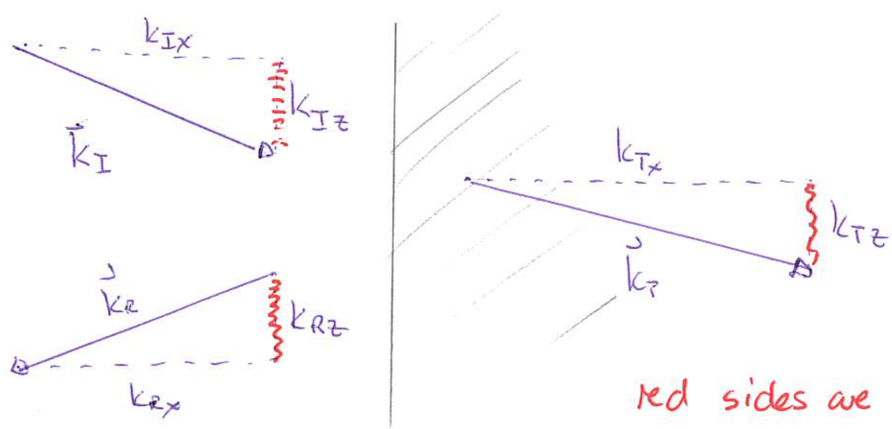
where $n_i = \frac{c}{v_i}$ and $v_i = 1/\sqrt{\mu_i \epsilon_i}$

Specifically if we define the xz plane to be that in which $\vec{k}_I$ resides. Then

$$\vec{k}_I = k_{Ix}\hat{x} + k_{Iz}\hat{z} \quad \Rightarrow \quad \vec{k}_R = k_{Rx}\hat{x} + k_{Rz}\hat{z}$$

$$\Rightarrow \quad \vec{k}_T = k_{Tx}\hat{x} + k_{Tz}\hat{z}$$

and $k_{Iz} = k_{Rz} = k_{Tz}$.



red sides are all same

Transmitted and reflected fields

The derivation of the geometric optics rules did not relate the fields on either side of the boundary. These are important for determining the intensities of the transmitted and reflected light. Note that since the intensities account for the energy transported by the wave, the intensities of the reflected and transmitted waves must be less than that of the incident wave.
How does the incident intensity divide?

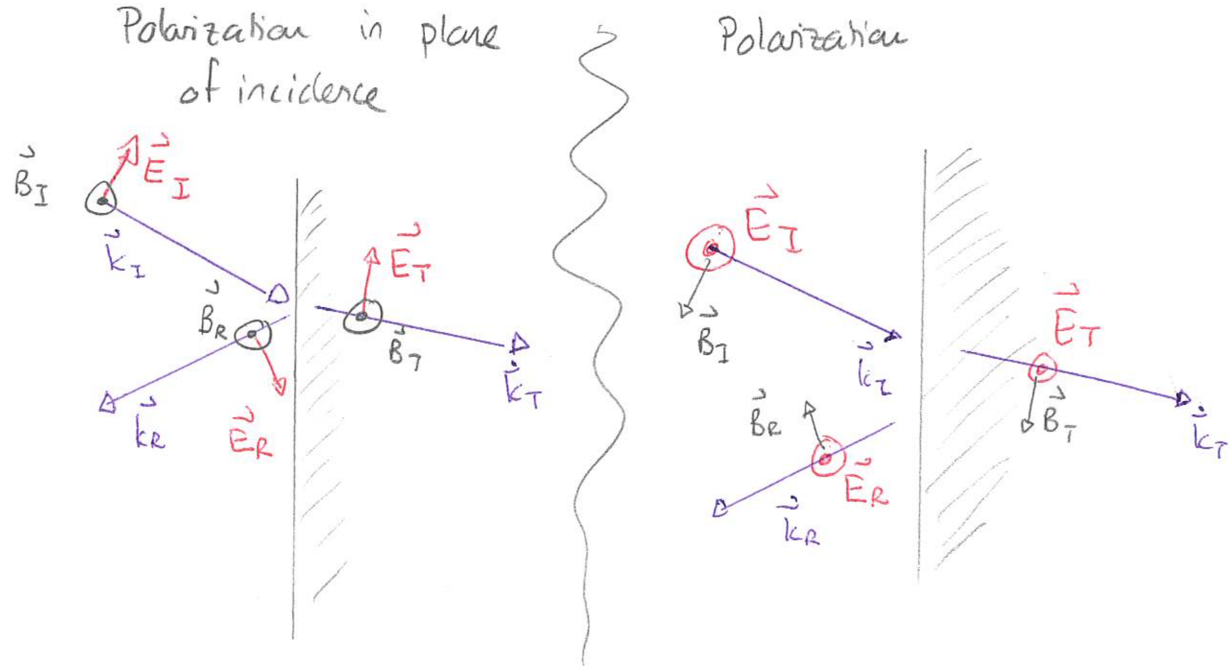
DEMO: PhET Bending light

The basic rules will be determined from the electric and magnetic field boundary conditions

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad B_1^{\perp} = B_2^{\perp}$$

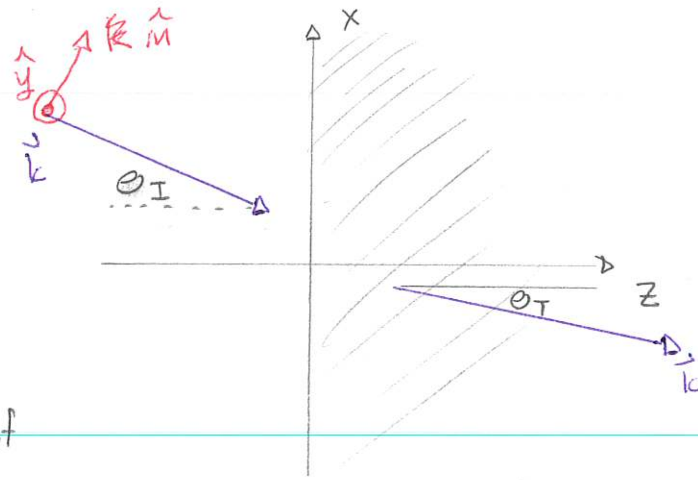
$$\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$$

It will emerge that the polarization of the incident wave will be important. The reflected and transmitted intensities will be different for the two extreme cases:



DEMO – create reflecting surface/interface
– illustrate $\vec{k}_I$ with rod – illustrate $\vec{E}_I, \vec{E}_R$ with arrows

We will cover both cases by decomposing the electric field vector into components along two directions:



* $\hat{y}$ (perpendicular to plane of incidence)

* ~~$\hat{m} = \hat{y} \times \hat{k}$ (parallel to plane of~~ incidence)

Note that since $\hat{k}$ is different for all three waves, the vector $\hat{m}$ will be different for all three directions. Thus there will be $\hat{m}_I, \hat{m}_R, \hat{m}_T$ in this case. Generically we will be able to express

$$\vec{E} = E_y \hat{y} + E_m \hat{m}$$

for each of the three fields $\vec{E}_I$, $\vec{E}_R$ and $\vec{E}_T$. We can do the same with the magnetic field always using.

$$\vec{B} = \frac{1}{v}(\hat{k} \times \vec{E})$$

It follows that

$$\vec{B} = \frac{1}{v}\left[E_y \, \hat{k} \times \hat{y} + E_m \, \hat{k} \times \hat{m}\right]$$

and $\hat{k} \times \hat{y} = -\hat{m}$ and $\hat{k} \times \hat{m} = \hat{y}$ gives

$$\vec{B} = \frac{1}{v}\left[E_m \hat{y} - E_y \hat{m}\right]$$

Since the vectors $\hat{m}$ differ for the three waves we need to express these in terms of $\hat{x}, \hat{y}, \hat{z}$.

1 Polarization unit vectors for reflection and transmission

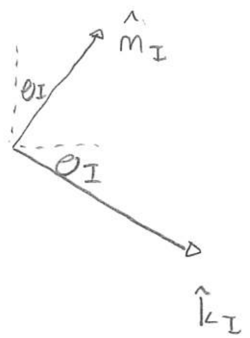
Consider three plane waves incident upon a surface at $z = 0$ with the xz-plane as the plane of incidence. Let θ_I be the angle between the incident and normal, with similar definitions for θ_T and θ_R. Let $\hat{\mathbf{m}} = \hat{\mathbf{y}} \times \hat{\mathbf{k}}$ be the unit vector in the plane of incidence.

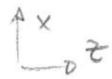
a) Determine an expression for $\hat{\mathbf{m}}_I$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{z}}$ and θ_I.

b) Determine an expression for $\hat{\mathbf{m}}_R$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{z}}$ and θ_R.

c) Determine an expression for $\hat{\mathbf{m}}_T$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{z}}$ and θ_T.

Answer: a)

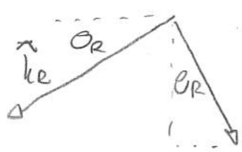




$$\hat{m}_I = \sin\theta_I \hat{z} + \cos\theta_I \hat{x}$$

$$\hat{m}_I = \cos\theta_I \hat{x} + \sin\theta_I \hat{z}$$

b)



$$\hat{m}_R = -\cos\theta_R \hat{x} + \sin\theta_R \hat{z}$$

c) Similar to incident $\quad \hat{m}_T = \cos\theta_T \hat{x} + \sin\theta_T \hat{z}$

Note that $\theta_R = \theta_I$ and thus

$$\boxed{\begin{aligned} \hat{m}_I &= \cos\theta_I \hat{x} + \sin\theta_I \hat{z} \\ \hat{m}_R &= -\cos\theta_I \hat{x} + \sin\theta_I \hat{z} \\ \hat{m}_T &= \cos\theta_T \hat{x} + \sin\theta_T \hat{z} \end{aligned}}$$

Now the three waves have electric fields:

1) incident $\vec{E}_I = \vec{E}_{0I}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}$

where $\boxed{\vec{E}_{0I} = E_{0Iy}\,\hat{y} + E_{0Im}\,\hat{m}_I}$

and E_{0Iy} and E_{0Im} are complex.

2) reflected $\vec{E}_R = \vec{E}_{0R}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}$

$$\boxed{\vec{E}_{0R} = E_{0Ry}\,\hat{y} + E_{0Rm}\,\hat{m}_R}$$

3) transmitted $\vec{E}_T = \vec{E}_{0T}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$

$$\boxed{\vec{E}_{0T} = E_{0Ty}\,\hat{y} + E_{0Tm}\,\hat{m}_T}$$

In all cases, at the boundary, $e^{i(\vec{k}_\bullet \cdot \vec{r} - \omega t)}$ are the same for all waves. Thus we can ignore these at the boundary and when comparing

$$\vec{E}_1 = \vec{E}_T + \vec{E}_R \qquad \text{to} \qquad \vec{E}_2 = \vec{E}_T$$

we only need to relate

$$\vec{E}_1 = \vec{E}_{0I} + \vec{E}_{0R} \qquad \text{to} \qquad \vec{E}_{0T} = \vec{E}_2$$

2 Electric field polarization components at the boundary.

The standard polarization unit vectors are $\hat{\mathbf{y}}$ and $\hat{\mathbf{m}} = \hat{\mathbf{y}} \times \hat{\mathbf{k}}$.

a) Determine an expression for $\mathbf{E}_1$ in terms of $E_{oIy}, E_{oIm}, E_{oRy}, E_{oRm}$, and standard Cartesian unit vectors.

b) Determine an expression for $\mathbf{E}_2$ in terms of E_{oTy}, E_{oTm} and standard Cartesian unit vectors.

c) Apply the electric field boundary matching conditions obtain linear relationships between E_{oTy}, E_{oTm} and $E_{oIy}, E_{oIm}, E_{oRy}, E_{oRm}$. Use

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}$$

$$\beta = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Answer. a) $\vec{E}_1 = \vec{E}_{oI} + \vec{E}_{oR}$

$$= E_{oIy}\hat{y} + E_{oIm}\hat{m}_I + E_{oRy}\hat{y} + E_{oRm}\hat{m}_R$$

$$= E_{oIy}\hat{y} + E_{oIm}(\cos\theta_I \hat{x} + \sin\theta_I \hat{z})$$
$$+ E_{oRy}\hat{y} + E_{oRm}(-\cos\theta_I \hat{x} + \sin\theta_I \hat{z})$$

$$\vec{E}_1 = \cos\theta_I (E_{oIm} - E_{oRm})\hat{x} + (E_{oIy} + E_{oRy})\hat{y} + \sin\theta_I (E_{oIm} + E_{oRm})\hat{z}$$

b) $\vec{E}_2 = E_{oTy}\hat{y} + E_{oTm}\hat{m}_T \Rightarrow \vec{E}_2 = E_{oTm}\cos\theta_T \hat{x} + E_{oTy}\hat{y} + E_{oTm}\sin\theta_T \hat{z}$

c) $\epsilon_1 E_1^\perp = \epsilon_2 E_2^\perp \Rightarrow E_{1z} = \frac{\epsilon_2}{\epsilon_1} E_{2z}$

$$\Rightarrow \sin\theta_I (E_{oIm} + E_{oRm}) = \frac{\epsilon_2}{\epsilon_1} \sin\theta_T E_{oTm}$$

$$\Rightarrow E_{oIm} + E_{oRm} = \frac{\epsilon_2}{\epsilon_1} \frac{\sin\theta_T}{\sin\theta_I} E_{oTm}$$

Now $n_2 \sin\theta_T = n_1 \sin\theta_I \Rightarrow \frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$

Then $\quad \epsilon\mu = \frac{1}{v^2} \Rightarrow \epsilon_1 = \frac{1}{\mu_1 v_1^2}$

So

$$E_{0IM} + E_{0RM} = \frac{\mu_1 v_1^2}{\mu_2 v_2^2}\frac{n_1}{n_2} E_{0TM}$$

But $\quad \frac{V_1}{V_2} = \frac{n_2}{n_1} \Rightarrow E_{0IM} + E_{0RM} = \frac{\mu_1}{\mu_2}\frac{n_2}{n_1} E_{0TM}$

$$\Rightarrow E_{0IM} + E_{0RM} = \beta E_{0TM}$$

Then $\quad \vec{E}_1^{\,\parallel} = \vec{E}_2^{\,\parallel} \Rightarrow E_{1x} = E_{2x} \Rightarrow \cos\theta_I (E_{0IM} - E_{0RM}) = \cos\theta_T E_{0TM}$

$$\Rightarrow E_{0IM} - E_{0RM} = \frac{\cos\theta_T}{\cos\theta_I} E_{0TM}$$

$$\Rightarrow E_{0IM} - E_{0RM} = \alpha E_{0TM}$$

Also $\quad E_{1y} = E_{2y} \Rightarrow E_{0Iy} + E_{0Ry} = E_{0Ty}$ ☒

The electric field matching conditions give three equations relating the six terms $E_{0Iy}, E_{0IM}, \ldots$

We can apply the magnetic field matching conditions. These require

$$\vec{B}_1 = \vec{B}_{0I} + \vec{B}_{0R}$$

$$= \frac{1}{V_1}(\hat{k}_I \times \vec{E}_{0I}) + \frac{1}{V_1}(\hat{k}_R \times \vec{E}_{0R})$$

$$= \frac{1}{V_1}\left[E_{0IM}\hat{y} - E_{0Iy}\hat{m} + E_{0RM}\hat{y} - E_{0Ry}\hat{y}\right]$$

$$= \frac{1}{V_1}\left[(E_{0Ry} - E_{0Iy})\cos\theta_I \hat{x} + (E_{0IM} - E_{0RM})\hat{y} - \sin\theta_I (E_{0Iy} + E_{0Ry})\hat{z}\right]$$

and

$$\vec{B}_2 = \frac{1}{V_2}\left[-E_{0Ty}\cos\theta_T \hat{x} + E_{0TM}\hat{y} - \sin\theta_T E_{0Ty}\hat{z}\right]$$

Then $\vec{B}_1^{\parallel} = \frac{\mu_1}{\mu_2} \vec{B}_2^{\parallel}$

$$\Rightarrow \quad (E_{0Ry} - E_{0Iy}) \cos\theta_I = -\frac{v_1}{v_2} E_{0Ty} \cos\theta_T \frac{\mu_1}{\mu_2}$$

$$\Rightarrow \quad E_{0Iy} - E_{0Ry} = \frac{v_1}{v_2} \frac{\cos\theta_T}{\cos\theta_I} E_{0Ty} \frac{\mu_1}{\mu_2}$$

$$E_{0Iy} - E_{0Ry} = \frac{\mu_1}{\mu_2} \frac{n_2}{n_1} \frac{\cos\theta_T}{\cos\theta_I} E_{0Ty}$$

The remaining magnetic field matching conditions reduce to those already obtain for electric fields. So we obtain four equations relating six quantities:

$$E_{0Iy} + E_{0Ry} = E_{0Ty}$$

$$E_{0Iy} - E_{0Ry} = \alpha\beta \, E_{0Ty}$$

$$E_{0IM} + E_{0RM} = \beta \, E_{0TM}$$

$$E_{0IM} - E_{0RM} = \alpha \, E_{0TM}$$

where

$$\alpha = \cos\theta_T / \cos\theta_I$$

$$\beta = \frac{\mu_1}{\mu_2} \frac{n_2}{n_1}$$

We can invert to solve for the transmitted and reflected fields in terms of the incident fields

3 Amplitudes of reflected and transmitted electric fields.

Using the components described in class

$$
\begin{aligned}
E_{oIy} + E_{oRy} &= E_{oTy} \\
E_{oIy} - E_{oRy} &= \alpha\beta E_{oTy} \\
E_{oIm} + E_{oRm} &= \beta E_{oTm} \\
E_{oIm} - E_{oRm} &= \alpha E_{oTm}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \frac{\cos\theta_T}{\cos\theta_I} \\
\beta &= \frac{\mu_1 n_2}{\mu_2 n_1}.
\end{aligned}
$$

a) Show that

$$
\begin{aligned}
E_{oRy} &= \frac{1-\alpha\beta}{1+\alpha\beta} E_{oIy} \\
E_{oTy} &= \frac{2}{1+\alpha\beta} E_{oIy} \\
E_{oRm} &= \frac{\beta-\alpha}{\beta+\alpha} E_{oIm} \\
E_{oTm} &= \frac{2}{\beta+\alpha} E_{oIm}.
\end{aligned}
$$

b) Determine an expression for α in terms of $\sin\theta_I$ and the indices of refraction.

c) Suppose that the electric field is polarized parallel to the plane of incidence. Show that there is an angle of incidence such that there is no reflected wave. Determine an expression for this angle in terms of n_1, n_2 and μ_1 and μ_2.

Answer: a) Add the first two $\Rightarrow$ $2E_{oIy} = (1+\alpha\beta) E_{oTy}$

$\Rightarrow$ $E_{oTy} = \frac{2}{1+\alpha\beta} E_{oIy}$

Then $E_{oRy} = E_{oTy} - E_{oIy} \Rightarrow E_{oRy} = \left(\frac{2}{1+\alpha\beta} - 1\right) E_{oIy}$

$\Rightarrow$ $E_{oRy} = \frac{1-\alpha\beta}{1+\alpha\beta} E_{oIy}$

Add the last two $\Rightarrow$ $E_{oTm} = \frac{2}{\alpha+\beta} E_{oIm}$

Then $E_{oRm} = \beta E_{oTm} - E_{oIm} \Rightarrow E_{oRm} = \left(\frac{2\beta}{\alpha+\beta} - 1\right) E_{oIm} = \left(\frac{\beta-\alpha}{\beta+\alpha}\right) E_{oIm}$

b) $\cos\theta_I = \sqrt{1-\sin^2\theta_I}$

$\cos\theta_T = \sqrt{1-\sin^2\theta_T}$

$n_2 \sin\theta_T = n_1 \sin\theta_I \Rightarrow \cos\theta_T = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_I}$

$$\Rightarrow \alpha = \left(\frac{1-\frac{n_1^2}{n_2^2}\sin^2\theta_I}{1-\sin^2\theta_I}\right)^{1/2}$$

c) In this case $E_{0Iy} = 0 \Rightarrow E_{0Ry} = E_{0Ty} = 0$ so the waves are all polarized in the plane of incidence.

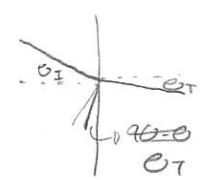
We need $E_{0RM} = 0 \Rightarrow \beta = \alpha$

$$\Rightarrow \frac{\mu_1}{\mu_2}\frac{n_2}{n_1} = \sqrt{\frac{1-\frac{n_1^2}{n_2^2}\sin^2\theta_I}{1-\sin^2\theta_I}} = \beta$$

$$\Rightarrow \beta^2(1-\sin^2\theta_I) = 1-\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_I$$

$$\Rightarrow \sin^2\theta_I\left[\left(\frac{n_1}{n_2}\right)^2-\beta^2\right] = 1-\beta^2$$

$$\Rightarrow \sin^2\theta_I = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2-\beta^2}$$



In general $\mu_1 \approx \mu_2$ and thus $\beta \approx \frac{n_2}{n_1}$

$$\Rightarrow \sin^2\theta_I \approx \frac{1-\beta^2}{1/\beta^2-\beta^2} = \frac{\beta^2(1-\beta^2)}{1-\beta^4} = \frac{\beta^2}{1+\beta^2}$$

Then $\tan^2\theta_I = \frac{\sin^2\theta_I}{\cos^2\theta_I} = \frac{\beta^2/1+\beta^2}{1-\beta^2/1+\beta^2} = \frac{\beta^2}{1}$

$$\Rightarrow \tan\theta_I \approx \beta \Rightarrow \tan\theta_I \approx \frac{n_2}{n_1}$$

will occur when

$\theta_I = 90° - \theta_T$

$\cos\theta_I = \sin\theta_T$

$n_? = \frac{n_1}{n_2}\sin\theta_I$

$\Rightarrow \tan\theta_I = \frac{n_2}{n_1}$