Fri: HW by 5pm   Tues: Read March 25   Tues March 25 HW 5pm

Phys 312 2025

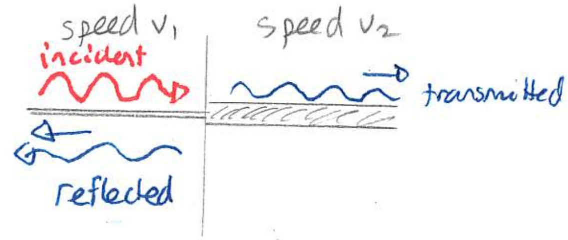
Lecture 15

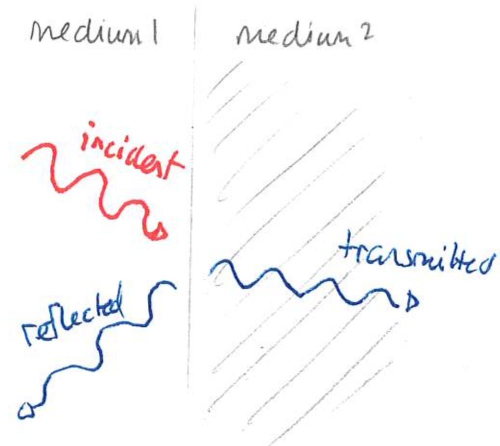
Reflection and transmission of electromagnetic waves

Waves that travel from one homogeneous medium to another will undergo reflection and transmission. The usual considerations will apply for this process for electromagnetic waves. There are two additional considerations:

1) electromagnetic waves can travel in three dimensions described by directions of propagation

2) electromagnetic waves are vectors (described by polarization).





We aim to use the basic rules of electromagnetism: (Maxwell's equations) boundary conditions) applied to sinusoidal waves to:

1) relate electric and magnetic fields of the transmitted and reflected waves to those of the incident wave
2) relate the directions of these waves
3) relate the intensities of these waves

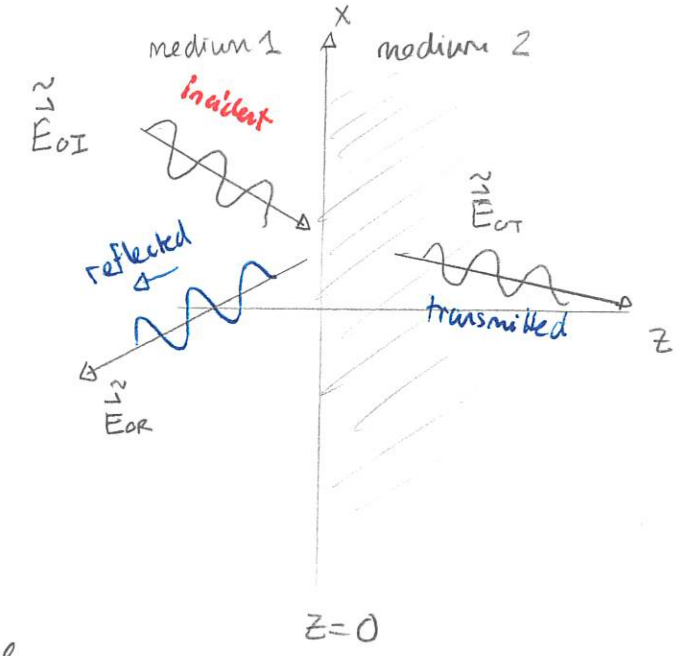
This is important for the understanding of:

1) reflection and transmission in optics
2) reflection of radio waves and microwaves.
3) electromagnetic waves and partly conducting surfaces.

We will consider the process for waves incident on a flat surface.

The general situation is analyzed in terms of electric fields and uses the complex representation. We will describe each wave using:

1) a wavenumber vector $\vec{k}_{+}$
2) a complex amplitude $\tilde{\vec{E}}_0$



In the set up we will have

1) the boundary is the $z=0$ plane
2) the incident wave is in the x-y plane.

Then the electric fields will be:

1) incident wave:

$$\tilde{\vec{E}}_I = \tilde{\vec{E}}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r} - \omega_I t)} \qquad \omega_I = k_I v_1 \qquad \tilde{\vec{B}}_I = \frac{1}{\omega_I}(\vec{k}_I \times \tilde{\vec{E}}_I)$$

2) reflected wave

$$\tilde{\vec{E}}_R = \tilde{\vec{E}}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega_R t)} \qquad \omega_R = k_R v_R \qquad \tilde{\vec{B}}_R = \frac{1}{\omega_R}(\vec{k}_R \times \tilde{\vec{E}}_R)$$

3) transmitted wave

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r} - \omega_T t)} \qquad \omega_T = k_T v_T \qquad \tilde{\vec{B}}_T = \frac{1}{\omega_T}(\vec{k}_T \times \tilde{\vec{E}}_T)$$

Based on introductory physics we would expect:

a) the frequencies are all the same: $\omega_I = \omega_R = \omega_T$

b) the directions of propagation all lie in the same plane

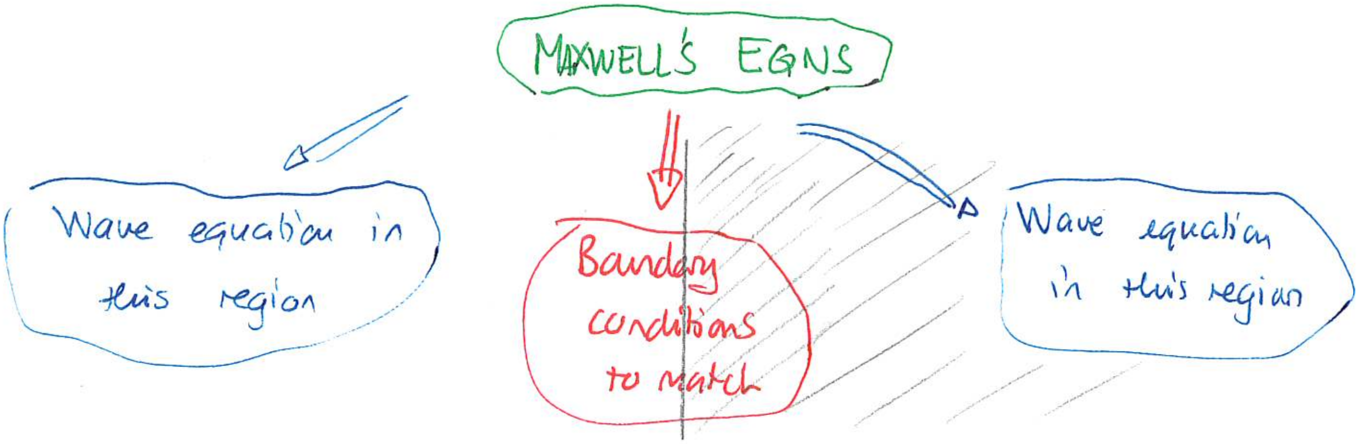
or $\vec{k}_I, \vec{k}_R, \vec{k}_T$

all lie in the xz plane

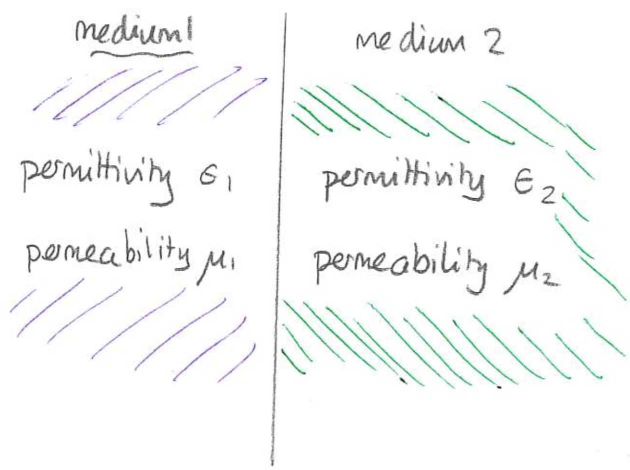
c) the law of reflection holds $k_{Rx} = -k_{Ix}$ and $k_{Rz} = -k_{RI}$

d) Snell's Law appears.

Rather than rely on given rules for transmission and reflection we want to use Maxwell's equations to obtain those rules



To make this feasible, we will assume that each medium is a dielectric/linear material and that there are no free charges. Then Maxwell's equations in a region with no free charges are:



$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

For linear media

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{H} = \frac{1}{\mu}\vec{B}$$

give

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

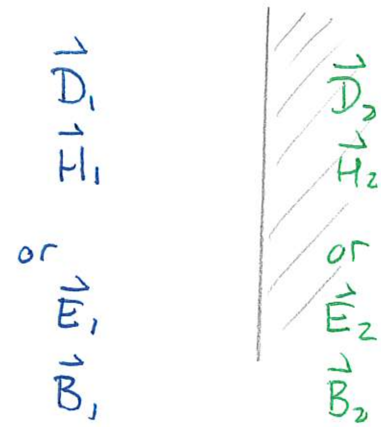
These give wave equations

$$\nabla^2\vec{E} = \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t}$$

where $\boxed{v = 1/\sqrt{\mu\epsilon}}$

The boundary conditions are obtained by the usual pillboxes and loops from Maxwell's equations. They give:

$$\left.\begin{aligned} D_1^{\perp} &= D_2^{\perp} \\ \vec{E}_1^{\parallel} &= \vec{E}_2^{\parallel} \\ B_1^{\perp} &= B_2^{\perp} \\ \vec{H}_1^{\parallel} &= \vec{H}_2^{\parallel} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} \epsilon_1 E_1^{\perp} &= \epsilon_2 E_2^{\perp} \\ \vec{E}_1^{\parallel} &= \vec{E}_2^{\parallel} \\ B_1^{\perp} &= B_2^{\perp} \\ \frac{1}{\mu_1}\vec{B}_1^{\parallel} &= \frac{1}{\mu_2}\vec{B}_2^{\parallel} \end{aligned}}$$



These will be our guiding equations to match fields across the boundary. We also need one mathematical result.

Lemma: Suppose that $A, B, C \neq 0$ and for all x

$$Ae^{\alpha x} + Be^{\beta x} = Ce^{\gamma x}$$

where α, β, γ are independent of x. Then:

1) $A + B = C$

2) $\alpha = \beta = \gamma$

Proof: 1) set $x = 0$. Then $A + B = C$

2) If $Ae^{\alpha x} + Be^{\beta x} = Ce^{\gamma x}$ then

$$Ae^{(\alpha-\gamma)x} + Be^{(\beta-\gamma)x} = C$$

Differentiate with respect to x $\Rightarrow (\alpha-\gamma)Ae^{(\alpha-\gamma)x} + (\beta-\gamma)Be^{(\beta-\gamma)x} = 0$

$\Rightarrow (\alpha-\gamma)Ae^{(\alpha-\beta)x} + (\beta-\gamma)B = 0$

Differentiate again $\Rightarrow (\alpha-\gamma)(\alpha-\beta)Ae^{(\alpha-\beta)x} = 0$

Set $x=0$. Then assuming $A \neq 0$ $(\alpha-\gamma)(\alpha-\beta) = 0 \Rightarrow \alpha = \gamma$ or $\alpha = \beta$.

If the first is true then $(\beta-\gamma)B = 0 \Rightarrow \beta = \gamma = \alpha$. A similar argument applies if $\alpha = \beta$.

Thus $\alpha = \gamma$ AND $\beta = \gamma$ $\blacksquare$

Applying the boundary conditions.

Given the proposed sinusoidal waves the electric field in medium 1 is

$$\tilde{\vec{E}}_1 = \tilde{\vec{E}}_I + \tilde{\vec{E}}_R$$

$$\tilde{\vec{E}}_1 = \tilde{\vec{E}}_{0I}\, e^{i(\vec{k}_I \cdot \vec{r} - \omega_I t)} + \tilde{\vec{E}}_{0R}\, e^{i(\vec{k}_R \cdot \vec{r} - \omega_R t)}$$

The electric field in medium 2 is

$$\tilde{\vec{E}}_2 = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_T \cdot \vec{r} - \omega_T t)}$$

The magnetic field in medium 1 is

$$\tilde{\vec{B}}_1 = \frac{1}{\omega_I}\left(\vec{k}_I \times \tilde{\vec{E}}_I\right) + \frac{1}{\omega_2}\left(\vec{k}_R \times \tilde{\vec{E}}_R\right)$$

The magnetic field in medium 2 is

$$\tilde{\vec{B}}_2 = \frac{1}{\omega_T}\left(\vec{k}_T \times \tilde{\vec{E}}_T\right)$$

One can show that the boundary matching conditions apply to complex representations just as well as to real representations of waves. Thus

$$\epsilon_1 \tilde{\vec{E}}_1^{\perp} = \epsilon_2 \tilde{\vec{E}}_2^{\perp}$$

$$\tilde{\vec{E}}_1^{\parallel} = \tilde{\vec{E}}_2^{\parallel}$$

etc...

Again we assume:

1) the boundary is $z=0$
2) for the incident wave $\vec{k}_I = k_{Ix}\hat{x} + k_{Iz}\hat{z}$

1 Electromagnetic wave matching conditions for the electric field.

Consider an electric field incident on a plane at $z = 0$. Suppose that the wavenumber vector for the incident field is in the xz plane.

a) Apply the boundary conditions to the electric fields at *one location on the boundary* and the lemma in class to relate the frequencies of the three waves.

b) Apply the boundary conditions to the electric fields at all locations along the boundary to show that all wavenumber vectors lie in the xz plane.

c) In the special case where the incident wave propagates perpendicular to the boundary, show that the wavenumber vectors of the reflected and transmitted waves are also perpendicular to to the boundary.

Answer: a) Consider $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$. Then $E_1^{\perp} = E_{1z}$

$E_2^{\perp} = E_{2z}$

$$\Rightarrow \epsilon_1 \tilde{E}_{0Iz}\, e^{i(\vec{k}_I\cdot\vec{r} - \omega_I t)} + \epsilon_1 E_{0Rz}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega_R t)}$$

$$= \epsilon_2 \tilde{E}_{0Tz}\, e^{i(\vec{k}_T\cdot\vec{r} - \omega_R t)}$$

Choosing $\vec{r} = 0\hat{x} + 0\hat{y} + 0\hat{z} \equiv 0$ gives:

$$\epsilon_1 \tilde{E}_{0Iz}\, e^{-i\omega_I t} + \epsilon_1 \tilde{E}_{0Rz}\, e^{-i\omega_R t} = \epsilon_2 \tilde{E}_{0Tz}\, e^{-i\omega_R t}$$

This can only be true for all t if

$$\omega_I = \omega_R = \omega_T$$

Thus the angular frequencies of the three waves are the same. Let

$$\omega = \omega_I = \omega_R = \omega_T$$

$$\Rightarrow \vec{E}_1 = \vec{E}_{0I}\, e^{i(\vec{k}_I\cdot\vec{r} - \omega t)} + \vec{E}_{0R}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega t)}$$

$$\vec{E}_2 = \vec{E}_{0T}\, e^{i(\vec{k}_T\cdot\vec{r} - \omega t)}$$

b) Use $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

$$\Rightarrow \quad \vec{E}_{0I}^{\parallel} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} + \vec{E}_{0R}^{\parallel} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} = \vec{E}_{0T}^{\parallel} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}$$

The ωt terms can all cancel. Then the given statement is true for all $\vec{r} = x\hat{x} + y\hat{y}$. Thus for all such $\vec{r}$

$$\vec{E}_{0I}^{\parallel} e^{i(\vec{k}_I \cdot \vec{r})} + \vec{E}_{0R}^{\parallel} e^{i(\vec{k}_R \cdot \vec{r})} = \vec{E}_{0T}^{\parallel} e^{i(\vec{k}_T \cdot \vec{r})}$$

$k_{Ix}x + k_{Iy}y$ $\qquad$ $k_{Rx}x + k_{Ry}y$ $\qquad$ $k_{Tx}x + k_{Ty}y$

These statements can only be true by the lemma if

$$k_{Ix} = k_{Rx} = k_{Tx}$$

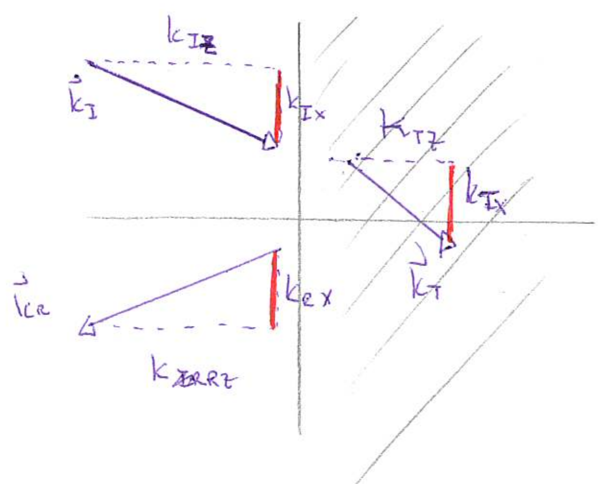
$$k_{Iy} = k_{Ry} = k_{Ty}$$

In our setup $k_{Iy} = 0 \Rightarrow k_{Ry} = k_{Ty} = 0$. Thus $\vec{k}_I, \vec{k}_R, \vec{k}_T$ all lie in the xz plane

$$\vec{k}_I = k_{Ix}\hat{x} + k_{Iz}\hat{z}$$

$$\vec{k}_R = k_{Rx}\hat{x} + k_{Rz}\hat{z}$$

$$\vec{k}_T = k_{Tx}\hat{x} + k_{Tz}\hat{z}$$



c) Then $k_{Ix} = 0$ and $k_{Rx} = k_{Tx} = 0$. The propagation vectors all lie along the z-axis.

The Maxwell's equations imply the boundary matching conditions and these imply:

For sinusoidal plane waves.

1) the frequencies of the transmitted, reflected and incident waves are the same

2) the wavenumber vectors all lie in one plane, defined by the wavenumber of the incident wave and the normal to the interface.

The plane in which the wavenumber all lie is called the <u>plane of incidence</u>. It is:

1) perpendicular to the surface that gives the interface
2) such that is contains $\vec{k}_I$.

Additionally the components of the wavevectors parallel to the interface are all equal. We can use this and

$$V = \omega / k$$

to relate the directions of the various waves

2 Directions of reflected and transmitted waves

a) Use the dispersion relation $\omega = kv$ to relate the magnitudes of the reflected wavenumber to the incident wavenumber. Repeat this for the transmitted wavenumber.

b) Use these to relate the z components of the wavenumber vectors, k_{Rz} and k_{Iz}. Sketch the directions of $\mathbf{k}_R$ and $\mathbf{k}_I$.

c) Derive the law of reflection from the previous result.

d) Relate the z components of the wavenumber vectors, k_{Tz} and k_{Iz}. Sketch the directions of $\mathbf{k}_T$ and $\mathbf{k}_I$ for the case where $v_2 < v_1$.

e) Derive Snell's law from the previous result.

Answer: a) For the reflected wave

$$V_1 = \omega / k_R$$

and for the incident wave

$$V_1 = \omega / k_I \quad \Rightarrow \quad k_I = \omega / V_1$$

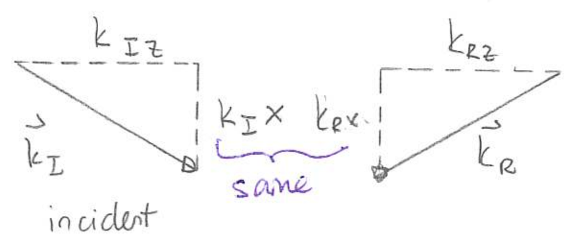
Thus $k_R = k_I$

For the transmitted wave

$$V_2 = \omega / k_T \quad \Rightarrow \quad k_T = \frac{\omega}{V_2}$$

Thus $$k_T / k_I = \frac{V_1}{V_2} \quad \Rightarrow \quad k_T = \frac{V_1}{V_2} k_I$$

b)

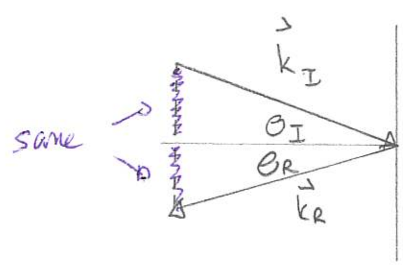


The only way that $k_R = k_I$ and $k_{Ix} = k_{Rx}$ is if

$$k_{Rz} = \pm k_{Iz}$$

Thus $k_{Rz} = k_{Iz}$

c)



geometry implies $\theta_I = \theta_R$

This is the law of reflection.

d) We have

$$k_{Tx} = k_{Ix}$$

$$k_T = k_I \frac{V_1}{V_2}$$

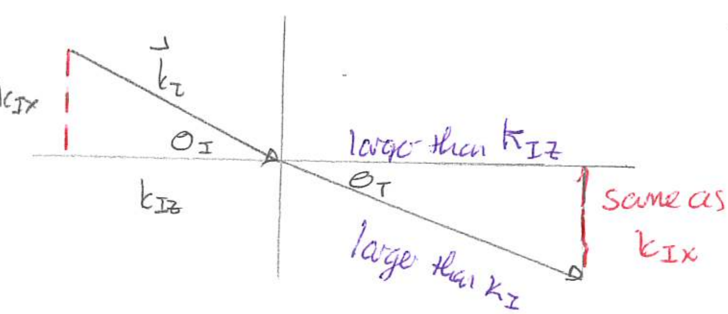
Then $$k_{Tz} = \sqrt{k_T^2 - k_{Tx}^2}$$

$$= \sqrt{k_I^2 \left(\frac{V_1}{V_2}\right)^2 - k_{Ix}^2}$$

$$= \sqrt{\left(k_{Ix}^2 + k_{Iz}^2\right)\left(\frac{V_1}{V_2}\right)^2 - k_{Ix}^2}$$

For the directions, if $V_2 < V_1$ then $k_T = \frac{V_1}{V_2} k_I > k_I$

Then the geometry gives the indicate direction (bends towards the normal)



$$k_T = k_I \frac{V_1}{V_2}$$

$$k_{Tx} = k_{Tx}$$

$$k_{Tx} = k_T \sin\theta_T$$

$$k_{Ix} = k_I \sin\theta_I$$

and $k_{Tx} = k_{Ix} \Rightarrow k_I \sin\theta_I = k_T \sin\theta_T$

Then $k_T = k_I \frac{V_1}{V_2}$ gives $\cancel{k_I} \sin\theta_I = \cancel{k_I} \frac{V_1}{V_2} \sin\theta_T$

$$\Rightarrow \frac{1}{V_1} \sin\theta_I = \frac{1}{V_2} \sin\theta_T$$

Defining the index of refraction to be $n = c/v$ gives

$$\frac{c}{V_1} \sin\theta_I = \frac{c}{V_2} \sin\theta_T \Rightarrow n_1 \sin\theta_I = n_2 \sin\theta_T$$

This is Snell's Law.

Thus:

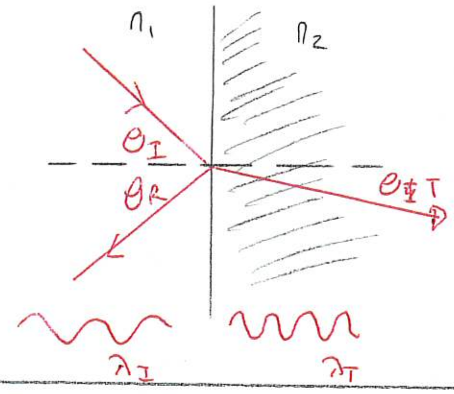
The electromagnetic boundary conditions imply that the directions of the reflected and transmitted waves obey

1) the law of reflection

$$\theta_I = \theta_R$$

2) Snell's law (law of refraction)

$$n_2 \sin\theta_T = n_1 \sin\theta_I$$



We can also see that

1) the frequencies of all waves are identical.

2) there is a wavelength shift across the boundary

$$k_T = k_I \frac{v_1}{v_2} \Rightarrow \frac{2\pi}{\lambda_T} = \frac{2\pi}{\lambda_I}\frac{v_1}{v_2}$$

$$\Rightarrow \boxed{\lambda_T = \lambda_I \frac{v_2}{v_1} \Rightarrow \lambda_T = \lambda_I \frac{n_1}{n_2}}$$