<u>Tues</u>: HW by 5pm <u>Thurs</u>: 9.3.1, 9.3.3

<u>Fri</u>: HW by 5pm

Sinusoidal electromagnetic waves

Maxwell's equations in a vacuum region can be manipulated to give wave equations:

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

Sinusoidal solutions to this have the form

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta) \quad \text{and similar for } \vec{B}$$

or

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where

$$\tilde{\vec{E}}_0 = \tilde{E}_{0x} e^{i\delta_x}\hat{x} + \tilde{E}_{0y} e^{i\delta_y}\hat{y} + \tilde{E}_{0z} e^{i\delta_z}\hat{z}$$

Previously we consider particular cases and found rules for the directions and magnitudes of the fields.

In general

1) The electric field is perpendicular to the direction of propagation

$$\vec{E}\cdot\vec{k} = 0$$

2) The magnetic field has the same wavenumber and frequency as the electric field

$$\vec{B} = \tilde{\vec{B}}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

3) The magnetic field is perpendicular to the direction of propagation.

$$\vec{B}\cdot\vec{k} = 0$$

4) The magnetic field is perpendicular to the electric field

$$\vec{B} = \frac{1}{\omega}(\vec{k}\times\vec{E})$$

which implies:

$$\vec{k} = \omega\,(\vec{E}\times\vec{B})/E^2$$

5) The magnitudes of the fields are related by

$$B = \frac{1}{c}E$$

The proofs are:

1) Direction of $\vec{E}$

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{\nabla}\cdot\left[\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \underbrace{\cancel{(\vec{\nabla}\cdot\tilde{\vec{E}}_0)}}_{\substack{0 \\ \text{constant}}} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{\vec{E}}_0\cdot\left[\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

Then $\vec{\nabla}\left[e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = i\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$\Rightarrow \quad i\,\tilde{\vec{E}}_0\cdot\vec{k}\, \cancel{e^{i(\vec{k}\cdot\vec{r}-\omega t)}} = 0 \quad \Rightarrow \quad \tilde{\vec{E}}_0\cdot\vec{k} = 0$$

$$\Rightarrow \quad \vec{E}\cdot\vec{k} = 0$$

2) Magnetic Field

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{\nabla}\times\left[\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow e^{i(\vec{k}\cdot\vec{r}-\omega t)} \underbrace{\cancel{\vec{\nabla}\times\tilde{\vec{E}}_0}}_{\substack{0 \\ \text{constant}}} - \tilde{\vec{E}}_0\times\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)} = -\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \frac{\partial\vec{B}}{\partial t} = -\tilde{\vec{E}}_0\times i\vec{k}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\left(\vec{k}\times\tilde{\vec{E}}_0\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

Integration gives $\quad \vec{B} = \frac{1}{\omega}\left(\vec{k}\times\tilde{\vec{E}}_0\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

We have the same wavenumber, the same frequency and

$$\tilde{\vec{B}} = \tilde{\vec{B}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

where $\quad \tilde{\vec{B}}_0 = \frac{1}{\omega}\left(\vec{k}\times\tilde{\vec{E}}_0\right)$

3) Direction of magnetic field

Using $\vec{\nabla}\cdot\vec{B} = 0$ and the same reasoning as for the electric field gives the result that $\vec{B}\cdot\vec{k} = 0$.

4) Magnetic and electric field

From $\tilde{\vec{B}}_0 = \frac{1}{\omega}\vec{k}\times\tilde{\vec{E}}_0$ we get

$$\vec{B} = \frac{1}{\omega}(\vec{k}\times\vec{E})$$

Then $\vec{E}\times\vec{B} = \frac{1}{\omega}\vec{E}\times(\vec{k}\times\vec{E})$

$$= \frac{1}{\omega}\vec{k}(\vec{E}\cdot\vec{E}) - \vec{E}\cancel{(\vec{k}\cdot\vec{E})}\ 0$$

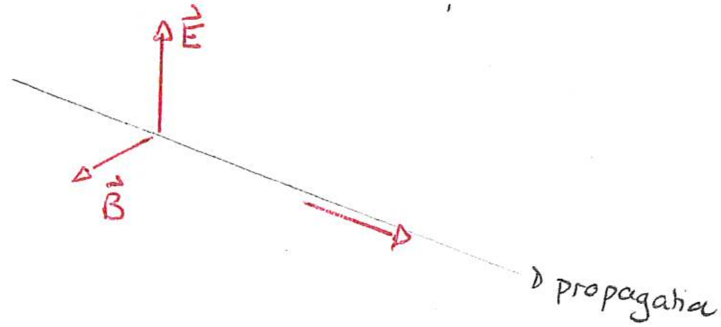
$$\Rightarrow \vec{k} = \omega(\vec{E}\times\vec{B})/E^2$$

5) Magnitudes

$$B = \frac{1}{\omega}kE\sin 90^\circ \Rightarrow B = \frac{k}{\omega}E \Rightarrow B = \frac{1}{c}E \quad \blacksquare$$

Thus we have:

The electromagnetic wave is completely specified by $\vec{k}$ and $\tilde{\vec{E}}_0$. These determine the electric and magnetic fields completely



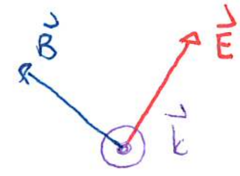
These statements will be true for any plane wave solution:

$$\vec{E}(\vec{r},t) = \tilde{\vec{E}}_0\, g(\vec{k}\cdot\vec{r}-\omega t)$$

Demo: PSU-S waves.

Polarization of electromagnetic waves

The electric and magnetic fields in electromagnetic waves are vectors and thus polarizations are possible. Consider waves that propagate along an axis out of the page. Both $\vec{E}$ and $\vec{B}$ are perpendicular.



Since $\vec{B}$ depends on $\vec{E}$ it suffices to consider just the electric field. In general

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

consider $x = y = z = 0$ (i.e. $\vec{r} = 0$). Then

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{-i\omega t}.$$

The only possibilities for $\tilde{\vec{E}}_0$ are in the xy plane (perpendicular to z)

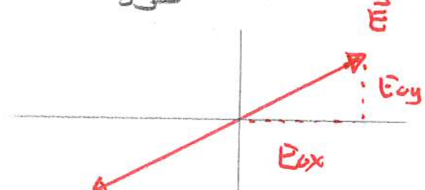
$$\tilde{\vec{E}}_0 = E_{0x} e^{i\delta_x}\hat{x} + E_{0y} e^{i\delta_y}\hat{y}$$

$$\Rightarrow \tilde{\vec{E}} = E_{0x} e^{i\delta_x} e^{-i\omega t}\hat{x} + E_{0y} e^{i\delta_y} e^{-i\omega t}\hat{y} = E_{0x} e^{i(-\omega t + \delta_x)}\hat{x} + E_{0y} e^{i(-\omega t + \delta_y)}\hat{y}$$

$$\Rightarrow \vec{E} = \mathrm{Re}(\tilde{\vec{E}}) = \boxed{E_{0x}\cos(\omega t - \delta_x)\hat{x} + E_{0y}\cos(\omega t - \delta_y)\hat{y} = \vec{E}}$$

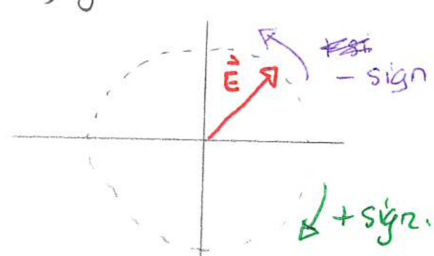
Possible states of polarization are:

1) linear polarization $\delta_x = \delta_y = \delta \Rightarrow \vec{E} = \cos(\omega t - \delta)[E_{0x}\hat{x} + E_{0y}\hat{y}]$



2) circular polarization $\delta_y = \delta_x \pm \pi/2$ and $E_{0x} = E_{0y} = E_0$

$$\Rightarrow \vec{E} = E_0\cos(\omega t - \delta_x)\hat{x} \mp E_0\sin(\omega t - \delta_x)\hat{y}$$



3) elliptical polarization If $\delta_y = \delta_x \pm \pi/2$ and $E_{0x} \neq E_{0y}$ then the electric field would describe an ellipse. This is an example of the most general case: elliptical polarization

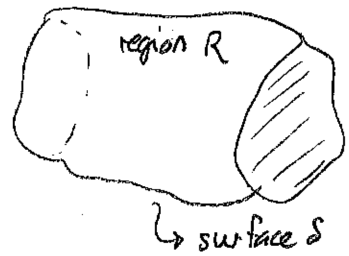
Energy and electromagnetic waves.

Electric and magnetic fields store and transport energy. This is described by a general result:

The total energy in any region

$$K+U$$

consists of the kinetic energy K of the particles in the region and the potential energy in the fields



$$U = \int_R \left[\frac{\epsilon_0}{2} \vec{E}\cdot\vec{E} + \frac{1}{2\mu_0} \vec{B}\cdot\vec{B} \right] d\tau$$

and satisfies

$$\frac{d}{dt}\left[K+U\right] = - \oint_S \vec{S}\cdot d\vec{a}$$

where the Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} \vec{E}\times\vec{B}$$

This describes the rate at which energy is transferred across the surface where $d\vec{a}$ is an outward normal.

In any vacuum region $K=0$ and thus the energy content is described entirely by U and the energy flow by the Poynting vector. This has the interpretation:

1) magnitude of $\vec{S}$ = energy flow per second per unit area
2) direction of $\vec{S}$ = direction of energy flow.

We consider this for sinusoidal plane electromagnetic waves.

1 Energy for electromagnetic plane waves

Consider plane waves that propagate along the $+z$ direction.

a) Show that the energy density is

$$u = \epsilon_0 \mathbf{E} \cdot \mathbf{E}.$$

b) Show that the Poynting vector is

$$\mathbf{S} = \epsilon_0 c E^2 \,\hat{\mathbf{z}} = uc\,\hat{\mathbf{z}}.$$

Answer: a)

$$u = \frac{\epsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{1}{2\mu_0} \vec{B} \cdot \vec{B} \quad \Rightarrow \quad u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$$

Then $B = \frac{1}{c} E$ gives:

$$u = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0 c^2} E^2$$

But $c = 1/\sqrt{\mu_0 \epsilon_0} \quad \Rightarrow \quad u = \epsilon_0 E^2$

b) $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$ and $\vec{B} = \frac{1}{\omega} (\vec{k} \times \vec{E})$ gives

$$\vec{S} = \frac{1}{\mu_0 \omega} \left[\vec{E} \times (\vec{k} \times \vec{E}) \right] = \frac{1}{\mu_0 \omega} \left[\vec{k} (\vec{E} \cdot \vec{E}) - \vec{E} (\vec{k} \cdot \vec{E}) \right]$$

But for electromagnetic plane waves $\vec{k} \cdot \vec{E} = 0$. Thus

$$\vec{S} = \frac{1}{\mu_0 \omega} E^2 \vec{k} = \frac{1}{\mu_0} \frac{k}{\omega} E^2 \hat{z} = \frac{1}{\mu_0 c} E^2 \hat{z}$$

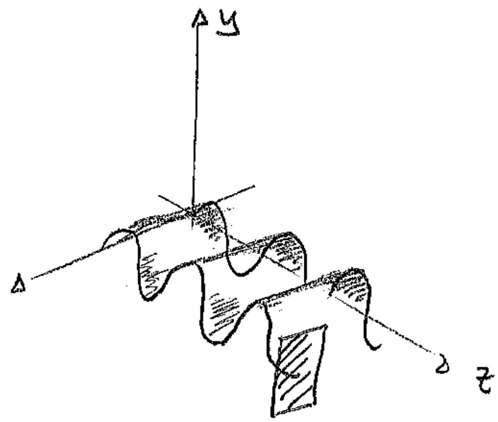
Then $c = 1/\sqrt{\mu_0 \epsilon_0} \Rightarrow \quad \mu_0 = \frac{1}{\epsilon_0 c^2}$

$$\Rightarrow \vec{S} = \epsilon_0 c E^2 \hat{z} = u\, \underbrace{c \hat{z}}_{\text{propagation speed and direction.}}$$

Intensity of electromagnetic radiation

We could attempt to determine the energy content of a sinusoidal plane wave but this has infinite extent and would therefore have infinite energy. Similarly the wavefronts have infinite extent and the rate at which energy would be transported would be infinite. We can, however, focus on a window of finite extent.

Suppose that we consider a window perpendicular to the direction of propagation, along $\hat{k}$. Then



$$\vec{S} = \epsilon_0 c E^2 \hat{k}$$

and

$$d\vec{a} = da\,\hat{k}$$

gives that the rate at which energy is transmitted through the window (the power) is

$$P = \int \vec{S}\cdot d\vec{a} = \int \epsilon_0 c E^2 da$$

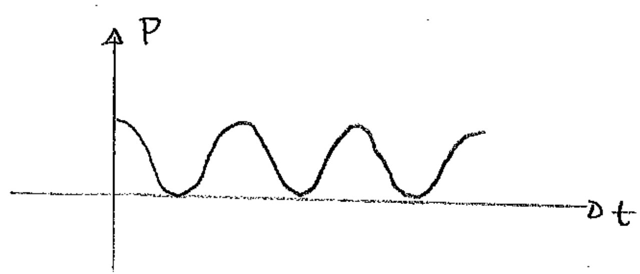
$$\Rightarrow \boxed{P = \epsilon_0 c E^2}$$

The general plane wave (propagating along $+\hat{z}$) is

$$\vec{E} = E_{0x}\cos(kz-\omega t+\delta_x)\,\hat{x} + E_{0y}\cos(kz-\omega t+\delta_y)\,\hat{y}$$

$$\Rightarrow \quad P = \epsilon_0 c\left[E_{0x}^2\cos^2(kz-\omega t+\delta_x) + E_{0y}^2\cos^2(kz-\omega t+\delta_y)\right]$$

This fluctuates with time, but is always positive. In some situations we might observe these time fluctuations. But in others, particularly optics they will be too rapid to observe.



We can determine and observe the average rate at which power flows. For a sinusoidally oscillating function we can average over one period, T.

Here $T = 2\pi/\omega$

Then

Given a function, f, with period $T = 2\pi/\omega$, the *time average* of f is:

$$\langle f \rangle := \frac{1}{T}\int_0^T f(t)\,dt$$

Then we define

The intensity of an electromagnetic wave is the time-averaged power per unit area:

$$I = \langle S \rangle$$

where S is the magnitude of the Poynting vector

Thus

For electromagnetic plane waves

$$I = \epsilon_0 c \langle E^2 \rangle$$

Then the power delivered to area A is

$$P = IA$$

2 Intensity for sinusoidal plane waves

Show that the intensity of any linearly polarized sinusoidal wave

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

is

$$I = \frac{1}{2} c\epsilon_0 E_0^2.$$

Answer:

$$S = \epsilon_0 c E^2$$

$$= \epsilon_0 c E_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\Rightarrow \quad \langle S \rangle = \epsilon_0 c E_0^2 \langle \cos^2(\vec{k}\cdot\vec{r} - \omega t)\rangle.$$

Then

$$\langle \cos^2(\vec{k}\cdot\vec{r} - \omega t)\rangle = \frac{1}{T}\int_0^T \cos^2(\vec{k}\cdot\vec{r} - \omega t)\, dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}\left[1 + \cos\left(2(\vec{k}\cdot\vec{r} - \omega t)\right)\right] dt$$

$$= \frac{1}{2T}\int_0^T dt + \frac{1}{2T}\int_0^T \cos\left[2(\vec{k}\cdot\vec{r} - \omega t)\right] dt$$

$$= \frac{1}{2} + \frac{1}{2T}\left(\frac{1}{-2\omega}\right) \sin\left[2(\vec{k}\cdot\vec{r} - \omega t)\right]_0^T$$

$$= \frac{1}{2} - \frac{1}{4\omega T}\left\{\sin(2\vec{k}\cdot\vec{r}) - \sin\left(2(\vec{k}\cdot\vec{r} - \omega T)\right)\right\}$$

$$= \frac{1}{2} - \frac{1}{4\omega T}\left\{\sin(2\vec{k}\cdot\vec{r}) - \sin(2\vec{k}\cdot\vec{r} - 4\pi)\right\}$$

cancel

$$= \frac{1}{2}$$

Thus $\langle S \rangle = \frac{1}{2} \Rightarrow \quad I = \frac{1}{2} c\epsilon_0 E_0^2$