Phys 312 2025

Lecture 11

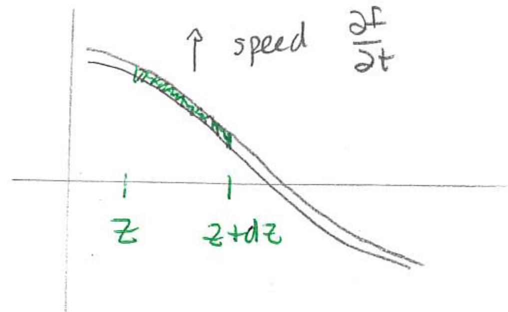
Tues: HW by 5pm

Read: 9.1.4, 9.2.1

Energy and power in waves

Consider a wave along a string. Since the string moves, there will be mass that moves and therefore kinetic energy

The segment from z to $z+dz$ has kinetic energy



$$dK = \frac{1}{2} dm \left(\frac{\partial f}{\partial t}\right)^2$$

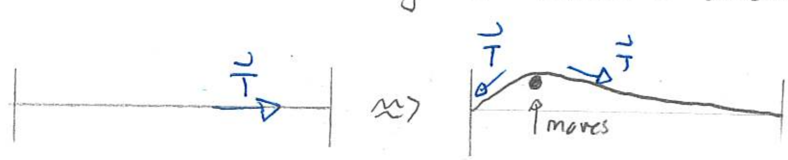
Then $dm = \mu dz$ where μ is the mass per unit length. Thus the kinetic energy in this section is

$$dK = \frac{1}{2}\mu \left(\frac{\partial f}{\partial t}\right)^2 dz$$

Then

> The kinetic energy in the region $a \leq z \leq b$ is
>
> $$K = \frac{1}{2}\int_a^b \underbrace{\mu \left(\frac{\partial f}{\partial t}\right)^2}_{\text{varies with } z} dz$$

There will have to be some form of potential energy associated with the deformation of the string and the work done against tension to deform it



We can show that the work and thus potential energy is given via.

The potential energy in the region $a \le z \le b$ is

$$U = \frac{1}{2}\int_a^b T\left(\frac{\partial f}{\partial z}\right)^2 dz$$

where T = tension in string.

Then we can combine these to show:

The energy in the region $a \le z \le b$ is

$$E = \frac{1}{2}\int_a^b \left[\mu\left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2\right] dz$$

and

$$\frac{dE}{dt} = -T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=a} + T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_b$$

quantify the rate at which energy enters at $z=a$ and leaves at $z=b$

<u>Proof</u>: $$\frac{dE}{dt} = \frac{1}{2}\int_a^b \left[\mu \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)^2 + T\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z}\right)^2\right] dz$$

$$= \frac{1}{2}\int_a^b \left[\mu\, 2\left(\frac{\partial f}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) + T2\frac{\partial f}{\partial z}\frac{\partial}{\partial t}\frac{\partial f}{\partial z}\right] dz$$

$$= \int_a^b \left[\mu \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial t^2} + T\frac{\partial f}{\partial z}\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial t}\right)\right] dz.$$

Then $$\int_a^b \underset{u}{\frac{\partial f}{\partial z}}\underbrace{\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial t}\right) dz}_{dv} = T\frac{\partial f}{\partial z}\frac{\partial f}{\partial t}\Big|_a^b - \int_a^b \frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial z^2} dz.$$

$$\Rightarrow \frac{dE}{dt} = \int_a^b \left[\mu\frac{\partial^2 f}{\partial t^2} - T\frac{\partial^2 f}{\partial z^2}\right]\frac{\partial f}{\partial t} dz - T\frac{\partial f}{\partial z}\frac{\partial f}{\partial t}\Big|_{z=a} + T\frac{\partial f}{\partial z}\frac{\partial f}{\partial t}\Big|_{z=b}.$$

But

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

and for waves on a string $v^2 = \frac{T}{\mu}$. So

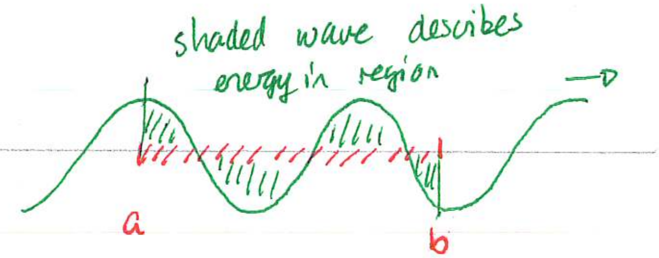
$$\frac{dE}{dt} = \int_a^b \underbrace{\cancel{\left[\mu \frac{\partial^2 f}{\partial t^2} - \mu \frac{\partial^2 f}{\partial z^2}\right]}}_{=0} \frac{\partial f}{\partial t}\, dz - \left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=a} + \left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=b}.$$

This proves the result $\boxtimes$

We can then interpret the remaining terms. Here $\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=b}$ can describe the rate at which energy leaves the region.

So we define

$$\boxed{\text{The rate at which energy flows from } z<z_0 \text{ to } z>z_0 \text{ is } \quad P_{z_0} = -T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=z_0}}$$



This is also the power transmitted from $z<z_0$ to $z>z_0$. Thus

$$\frac{dE}{dt} = P_a - P_b$$

2 Energy and power for one dimensional waves

Consider two sinusoidal waves

$$f_1(z,t) = A\cos(kz - \omega t)$$
$$f_2(z,t) = A\cos(kz + \omega t)$$

where $k > 0$ and $\omega > 0$. Assume that these are waves on a string, for which $v^2 = T/\mu$.

a) Determine the energy stored over the span of a single wavelength for each wave.

b) Determine the power delivered at any point. Does it depend on the location and the time? What does this power indicate about the direction in which each wave transports energy?

Answer: a)

$$E = \frac{1}{2}\int_{z_0}^{z_0+\lambda} \mu\left(\frac{\partial f}{\partial t}\right)^2 dz + \frac{1}{2}\int_{z_0}^{z_0+\lambda} T\left(\frac{\partial f}{\partial z}\right)^2 dz$$

$$= \frac{\mu}{2}\int_{z_0}^{z_0+\lambda} \left[\mp\omega A\sin(kz \mp \omega t)\right]^2 dz + \frac{T}{2}\int_{z_0}^{z_0+\lambda}\left[- kA\sin(kz\mp\omega t)\right]^2 dz.$$

$$= \frac{1}{2}A^2\left(\omega^2\mu + Tk^2\right)\int_{z_0}^{z_0+\lambda} \sin^2(kz\mp\omega t)dz \qquad 1-2\sin^2\theta = \cos 2\theta$$

$$= \frac{A^2}{2}\left(\omega^2\mu + \mu v^2 k^2\right)\int_{z_0}^{z_0+\lambda}\frac{1}{2}\left(1 - \cos(2kz \mp 2\omega t)\right) dz.$$

The cos integrates to zero. Thus

$$E = \frac{A^2}{2}\mu(\omega^2 + \omega^2)\frac{1}{2}\lambda \qquad = \frac{1}{2}\mu\omega^2\lambda A^2$$

Now $\lambda = 2\pi/k \Rightarrow E = \mu\omega^2 k\pi A^2 \Rightarrow \boxed{E = \mu v\,\omega\pi A^2}$

b) $P = -\,T\left(\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial z}$

For $f_1(z,t) = A\cos(kz - \omega t)$

$$P = -T(-\omega)A(-\sin(kz-\omega t))\,A(-\sin(kz-\omega t))\,k.$$

$$= T\omega A^2 k \sin^2(kz - \omega t)$$

For $f_2 = A\cos(kz+\omega t)$ we will get

$$P = -T\omega A^2 k \sin^2(kz-\omega t)$$

Then $T = \mu v^2$ and $k = \omega/v$ gives

For right moving wave $f(z,t) = A\cos(kz-\omega t)$

$$P = \mu v \omega^2 A^2 \sin^2(kz-\omega t) > 0$$ $\Rightarrow$ energy flows right

For a left moving wave $f(z,t) = A\cos(kz+\omega t)$

$$P = -\mu v \omega^2 A^2 \sin^2(kz-\omega t) < 0$$ $\Rightarrow$ energy flows left.

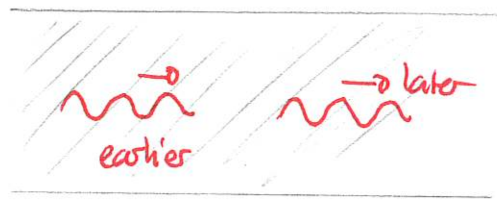
Both of these fluctuate with time and depend on position.

Also note that:

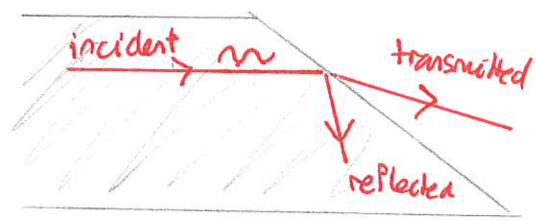
1) power is proportional to amplitude squared
2) power is proportional to wave speed.

Reflection and transmission of waves

In a homogeneous medium a wave travels with the same speed at all locations. A wave that propagates along one direction will continue to do so with unchanged speed and we observe this routinely in optics.



When the medium is not homogeneous will will observe partial transmission and reflection. We aim to describe this in general for waves. The question is



> Given a wave that is incident on a boundary:
>
> 1) in which directions are waves reflected and transmitted
> 2) what are the rates at which energy is reflected and transmitted

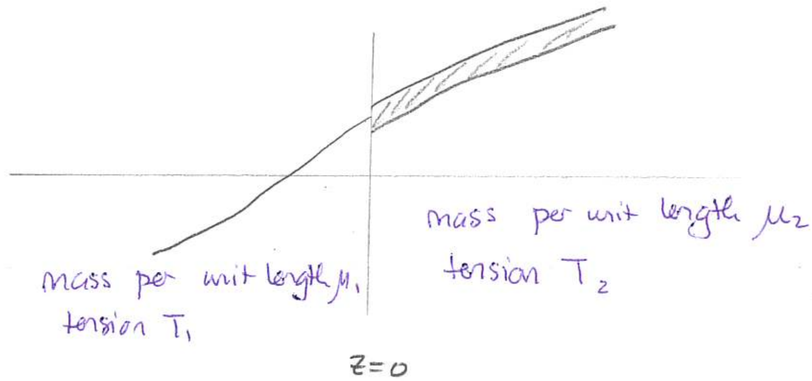
DEMO: PhET W.o.a.S

- Pulse reflected from fixed end
- Pulse reflected " open end
- Oscillate reflected " either

We will consider traveling sinusoidal waves and focus on the power delivered. Then

> For a wave $f(z,t) = A\cos(kz \mp \omega t)$ the power delivered is
>
> $$P = \pm T\omega k\, A^2 \sin^2(kz \mp \omega t)$$
>
> $$= \pm \frac{T\omega^2}{v} A^2 \sin^2(kz \mp \omega t)$$
>
> $$= \pm \mu v \omega^2 A \sin^2(kz \mp \omega t)$$

We will consider a medium consisting of two regions with different wavespeeds. The junction will be at $z=0$.



speed $v_1 = \sqrt{\frac{T_1}{\mu_1}}$ speed $v_2 = \sqrt{\frac{T_2}{\mu_2}}$

The strategy is:

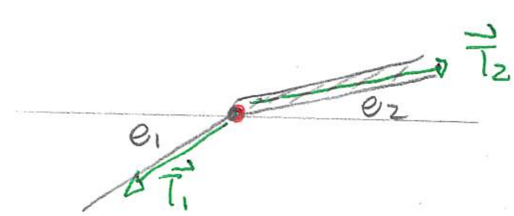
* solve the wave equation separately in each region
* match the solutions appropriately at the boundary.

In this case we get

	Left region $(z<0)$		Right region $(z>0)$	
wave eqn:	$\frac{\partial^2 f_1}{\partial z^2} = \frac{1}{v_1^2}\frac{\partial^2 f_1}{\partial t^2}$		$\frac{\partial^2 f_2}{\partial z^2} = \frac{1}{v_2^2}\frac{\partial^2 f_2}{\partial t^2}$	
solution:	$f_1(z,t)$	$z \leq 0$	$f_2(z,t)$	$z \geq 0$
unbroken string assumption	$f_1(0,t) = f_2(0,t)$ at all times			

This gives one boundary matching condition. The second boundary matching condition is obtained by considering the forces acting on an infinitesimally small string segment at $z=0$. We will show that the slopes of the string on either side must be equal. The first step here is that, on the segment

$$\vec{F}_{net} = m\vec{a}$$



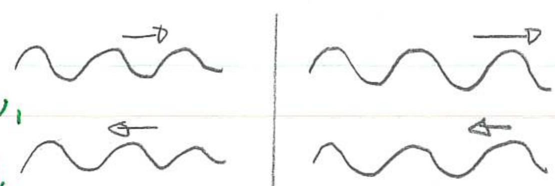
But $\vec{a} = 0$. Thus $\vec{T}_2 = -\vec{T}_1$. Then, the tension is tangent to the string. This gives:

$$\left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0}$$

Thus

> If the only discontinuity in the string is a jump in the mass per unit length at $z=0$ then
>
> $$f_1(0,t) = f_2(0,t)$$
>
> $$\left.\frac{\partial f_1}{\partial z}\right|_0 = \left.\frac{\partial f_2}{\partial z}\right|_0$$

We now apply these to complex representations of the waves. We consider four simultaneous possibilities provided that the frequency is the same on either side



on left

$$f_1(z,t) = \underbrace{A_1 e^{i(k_1 z - \omega t)}}_{} + \underbrace{B_1 e^{-i(k_1 z + \omega t)}}_{} \qquad f_2(z,t) = \underbrace{B_2 e^{i(k_2 z - \omega t)}}_{} + \underbrace{A_2 e^{-i(k_2 z + \omega t)}}_{}$$

travels right	travels left	travels right	travels ~~right~~ left
⇒ incident from left	⇒ reflected from left or ~~in~~ transmitted from right	⇒ reflected from right / transmitted from left	incident

We aim to relate B_1 and B_2 to A_1 and A_2

We need to invert

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1} = \frac{1}{k_2 + k_1} \begin{pmatrix} k_2 & 1 \\ -k_1 & 1 \end{pmatrix}$$

Thus

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \frac{1}{k_1+k_2} \begin{pmatrix} k_2 & 1 \\ -k_1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$= \frac{1}{k_1+k_2} \begin{pmatrix} k_1-k_2 & 2k_2 \\ 2k_1 & k_2-k_1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

$$\Rightarrow \quad B_1 = \frac{(k_1-k_2)}{(k_1+k_2)} A_1 + \frac{2k_2}{(k_1+k_2)} A_2$$

$$B_2 = \frac{2k_1}{k_1+k_2} A_1 + \frac{k_2-k_1}{k_1+k_2} A_2$$

Thus if

If two waves match at $z=0$ then

$$f_1(z,t) = A_1 e^{i(k_1 z - \omega t)} + B_1 e^{-i(k_1 z + \omega t)}$$

$$f_2(z,t) = B_2 e^{i(k_2 z - \omega t)} + A_2 e^{-i(k_2 z - \omega t)}$$

satisfy

$$B_1 = \frac{k_1-k_2}{k_1+k_2} A_1 + \frac{2k_2}{k_1+k_2} A_2$$

$$B_2 = \frac{2k_1}{k_1+k_2} A_1 + \frac{k_2-k_1}{k_1+k_2} A_2$$

2 Boundary matching conditions for a string

a) Apply the matching conditions to relate A_1, A_2, B_1 and B_2.

b) Express the matching conditions using matrices:

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

c) The inverse of a matrix that has non-zero determinant is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Use this and the previous result to determine expressions for B_1 and B_2 in terms of A_1 and A_2.

Answer: a) $f_1(0,t) = f_2(0,t) \Rightarrow A_1 e^{-i\omega t} + B_1 e^{-i\omega t} = A_2 e^{-i\omega t} + B_2 e^{-i\omega t}$

$\Rightarrow A_1 + B_1 = A_2 + B_2$

$\left.\frac{\partial f_1}{\partial z}\right|_{z=0} = \left.\frac{\partial f_2}{\partial z}\right|_{z=0} \Rightarrow ik_1 A_1 e^{-i\omega t} - ik_1 B_1 e^{-i\omega t} = ik_2 B_2 e^{-i\omega t} - ik_2 A_2 e^{-i\omega t}$

$\Rightarrow k_1 A_1 - k_1 B_1 = k_2 A_2 - k_2 B_2$

b) $B_1 - B_2 = A_2 - A_1$

$+k_1 B_1 + k_2 B_2 = k_2 A_2 + k_1 A_1$

$$\Rightarrow \begin{pmatrix} 1 & -1 \\ +k_1 & k_2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} -1 & +1 \\ +k_1 & k_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ k_1 & k_2 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 1 \\ k_1 & k_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$