Phys 312 2025

Lecture 10

Fri: HW by 5pm

Tues: Exam I in class

Covers Ch 4, 6, 8.1

HW 1 $\to$ 9

Bring: * ½ letter sheet single sided

* given - copies of front/back covers of text

## Waves in one dimension

Waves in one dimension can be described by:

* $f(z,t)$ - a function that describes disturbance (e.g. displacement) away from equilibrium where

* $z$ = location in one dimension

* $t$ = time

The classical wave equation is:

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

where $v$ is the wave speed.

The general solution to this is

$$f(z,t) = g(z-vt) + h(z+vt)$$

where $g$ and $h$ are arbitrary differentiable functions of one variable. We interpret

$g(z-vt)$ $\rightsquigarrow$ disturbance that propagates to right ($z$ increasing)

$h(z+vt)$ $\rightsquigarrow$ " " " " left ($z$ decreasing)

## Sinusoidal waves

Although a generic form of the solution to the wave equation exists, it is useful to consider a special class of solution from which all other solutions can be constructed. These are sinusoidal waves.

A sinusoidal wave has form

$$f(z,t) = A\cos[k(z-vt)+\delta]$$

where $A, k, \delta$ are constants

These can be combined by superposition to give general solutions

$$f(z,t) \sim \sum_{\substack{\text{all values} \\ \text{of } k}} A(k)\cos[k(z-vt)+\delta(k)]$$

The basic sinusoidal wave can be expressed as

$$f(z,t) = A\cos(kz-\omega t+\delta)$$

where $\omega = kv$ (the dispersion relation).

The constants have interpretation:

1) Amplitude $A$ = maximum displacement of the disturbance

2) Wavenumber $k$ describes the rate at which the wave repeats itself. To be specific, if $z' = z + 2\pi/k$ then

$$f(z',t) = A\cos[kz - kvt + \cancel{k}2\pi/\cancel{k} + \delta] = A\cos[k(z-vt)+\delta]$$
$$= f(z,t)$$

This is also the shortest increment such that this occurs. In general

$$f(z',t) = f(z,t) \iff A\cos\left[k(z'-vt)+\delta\right] = A\cos\left[k(z-vt)+\delta\right]$$

$$\iff \quad kz' = kz + 2\pi n \qquad \text{where } n \text{ is an integer}$$

$$\Rightarrow \quad z' = z + 2\pi n/k$$

Thus the smallest non-trivial increment is $\Delta z = 2\pi/k$. This is called the wavelength of the wave

$$\lambda = 2\pi/k \quad \Rightarrow \quad k = 2\pi/\lambda$$

$\rightsquigarrow$ wave repeats every increment $\Delta z = \lambda$

3) <u>Angular frequency</u>, $\omega$, describes the rate at which the wave repeats in time. Again let $T$ be the shortest time in which the wave repeats. So for all $z$

$$f(z,t+T) = f(z,t)$$

Thus

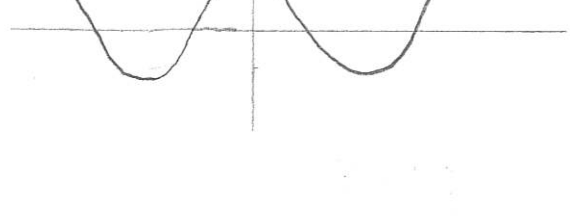
$$A\cos\left[kz-\omega(t+T)+\delta\right] = A\cos\left[kz-\omega t+\delta\right]$$

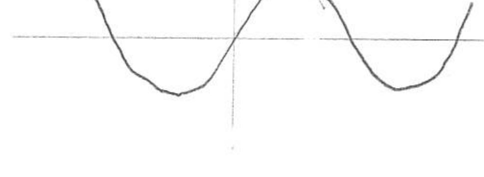
$$\iff \cos\left[kz-\omega t+\delta-\omega T\right] = \cos\left[kz-\omega t+\delta\right]$$

$$\iff \quad \omega T = 2\pi n \quad \text{for all integers } n.$$

Then the shortest time is $T = 2\pi/\omega$. The frequency is defined as $f = 1/T \Rightarrow f = \omega/2\pi \Rightarrow \omega = 2\pi f$

4) <u>Phase</u>, $\delta$ establishes the displacement and transverse velocity at $z=0$ and $t=0$



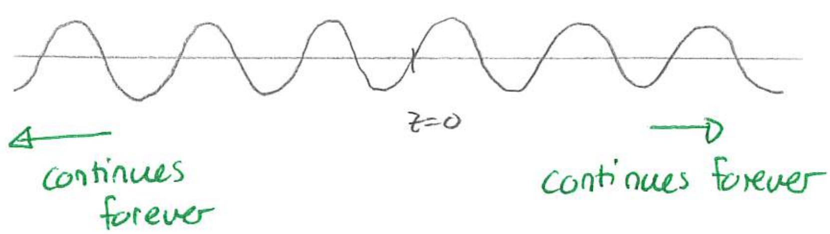
In general we choose $\omega$ and $k$ positive. Then

$$\boxed{\begin{array}{lll} f(z,t) = A\cos(kz-\omega t+\delta) & \Rightarrow \text{wave travels right (along } +z) & \text{speed } v = \omega/k \\ f(z,t) = A\cos(kz+\omega t+\delta) & \Rightarrow \text{wave travels left (along } -z) & \text{speed } v = \omega/k \end{array}}$$

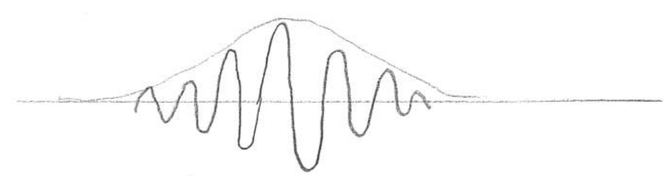
Note that these waves extend infinitely along all $z$, that is for

$$-\infty < z < \infty$$

In this sense they are a mathematical idealization.




In practice, one can describe localized versions of such waves by forming appropriate superpositions of sinusoidal waves with appropriate amplitudes that depend on $k$.



$$\hookrightarrow \sum_k A(k)\cos[kz-\omega t]$$

# Complex Representation of Waves

Representing sinusoidal waves in terms of trigonometric functions is less convenient than representing them in terms of complex exponentials. Recall that complex numbers have the form:

$$z = u + iv$$

where $u, v$ are real. Then the real part of $z$ is

$$\mathrm{Re}[z] = u$$

and the imaginary part of $z$ is

$$\mathrm{Im}[z] = v$$

An important definition is the complex exponential. For any real $\theta$,

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta}$$

Now if $A$ is real we consider

$$\tilde{f}(z,t) = Ae^{i(kz - \omega t + \delta)}$$

Then

$$\mathrm{Re}[\tilde{f}(z,t)] = A\cos(kz - \omega t + \delta) = f(z,t)$$

So the real part of $\tilde{f}(z,t)$ is a sinusoidal wave. We can thus use $\tilde{f}(z,t)$ to represent a wave with the understanding that the physical displacement is represented by $\mathrm{Re}[\tilde{f}(z,t)]$. When there are operations involving more than one wave, it is usually more convenient to use the complex representations to do the operations, and eventually extract the real part.

Note that

$$\tilde{f}(z,t) = Ae^{i\delta} e^{i(kz-\omega t)}$$

and if we define a complex amplitude $\tilde{A} = Ae^{i\delta}$ then

$$\tilde{f}(z,t) = \tilde{A} e^{i(kz-\omega t)}$$

so we have the following scheme:

Real wave equation

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Complex wave equation

$$\frac{\partial^2 \tilde{f}}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \tilde{f}}{\partial t^2}$$

Real sinusoidal solns:

travel along $+\hat{z}$ $\Big\}$ $f(z,t) = A\cos(kz-\omega t+\delta)$

travel along $-\hat{z}$ $\Big\}$ $f(z,t) = A\cos(kz+\omega t+\delta)$

Complex exponential solutions

along $+\hat{z}$ $\quad \tilde{f}(z,t) = \tilde{A}\, e^{i(kz-\omega t)}$

along $-\hat{z}$ $\quad \tilde{f}(z,t) = \tilde{A}\, e^{i(kz+\omega t)}$

Do math here.

same solution

$$f(z,t) = \text{Re}\left[\tilde{f}(z,t)\right]$$

When finished translate here to get physical solutions.

## 1 Superpositions of solutions via complex exponentials

Consider two solutions

$$f_1(z,t) = A\cos(k_1 z - \omega_1 t)$$
$$f_2(z,t) = A\cos(k_2 z - \omega_2 t)$$

The aim of this exercise is to determine an expression for the superposition

$$f(z,t) = f_1(z,t) + f_2(z,t).$$

a) Let $\Delta k = k_2 - k_1$. Use this and $v = \omega/k$ to rewrite $f_i(z,t)$ in terms of $k_1, \Delta k$ and $v$ rather than $k_2, \omega_1, \omega_2$.

b) Express each of $f_i(z,t)$ as a complex solution and use the superposition of the complex solutions to determine an expression for $f(z,t)$ as a product of sinusoidal functions. You will have to use

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta.$$

Answer: a) $f_1(z,t) = A\cos(k_1 z - v k_1 t)$

$$f_2(z,t) = A\cos((k_1+\Delta k)z - v(k_1+\Delta k)t)$$

$$= A\cos[k_1 z - v k_1 t + \Delta k z - v\Delta k t]$$

b) $\tilde{f}_1(z,t) = A e^{i(k_1 z - v k_1 t)}$

$$\tilde{f}_2(z,t) = A e^{i(k_1 z - v k_1 t + \Delta k(z-vt))}$$

$$\tilde{f}(z,t) = A e^{i(k_1 z - v k_1 t)}\left[1 + e^{i\Delta k(z-vt)}\right]$$

$$= A e^{i(k_1 z - v k_1 t)} e^{i\Delta k(z-vt)/2}\underbrace{\left[e^{i\frac{\Delta k(z-vt)}{2}} + e^{-i\frac{\Delta k(z-vt)}{2}}\right]}_{2\cos\left[\frac{\Delta k(z-vt)}{2}\right]}$$

$$= 2A\cos[\Delta k(z-vt)]\, e^{i[(k_1+\Delta k/2)(z-vt)]}$$

Taking the real part gives

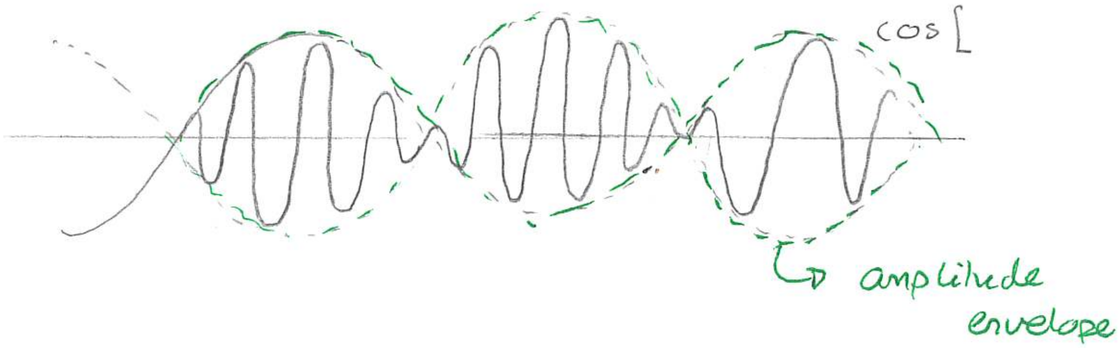
$$f(z,t) = 2A\cos\left[\frac{\Delta k(z-vt)}{2}\right]\cos[(k_1+\Delta k/2)(z-vt)].$$

This represents a traveling wave that can be thought of as

↳ a wave with wavenumber $k_1 + \frac{\Delta k}{2} = \frac{k_2 + k_1}{2}$ smaller wavelength

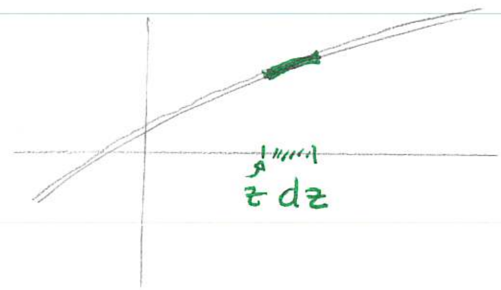
↳ an amplitude envelope with wavenumber

$\frac{\Delta k}{2} = \frac{k_2 - k_1}{2}$ larger wavelength



# Energy in waves

Consider a wave on a string. Since the string has mass, every segment will have kinetic energy. Let $\mu$ be the mass per unit length. Then the shaded segment has mass




$$dm = \mu dz$$

The speed of this section is $\frac{\partial f}{\partial t}$ and thus the section has kinetic energy

$$dK = \frac{1}{2}\mu dz \left(\frac{\partial f}{\partial t}\right)^2$$

Then the kinetic energy in the region $a \le z \le b$ is

$$K = \frac{1}{2}\int_a^b \mu \left(\frac{\partial f}{\partial t}\right)^2 dz$$

There will presumably be a potential energy. It will emerge that if $T$ is the tension in the string, then the potential energy for $a \le z \le b$ is

$$U = \frac{1}{2}\int_a^b T \left(\frac{\partial f}{\partial z}\right)^2 dz$$

We will show:

In the region $a \le z \le b$ the total energy in the wave is

$$\boxed{E = \frac{1}{2}\int_a^b \left\{ \mu \left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2 \right\} dz}$$

and

$$\boxed{\frac{dE}{dt} = \underbrace{-T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=a}}_{\text{rate at which energy enters } z=a} + \underbrace{T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)_{z=b}}_{\text{rate at which energy leaves } z=b}}$$

Proof:

$$\frac{dE}{dt} = \frac{1}{2}\int_a^b \mu \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right)^2 dz + \frac{1}{2}\int_a^b T \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial z}\right)^2 dz$$

$$= \int_a^b \mu \left(\frac{\partial f}{\partial t}\right)\frac{\partial^2 f}{\partial t^2} dz + \int_a^b T \underbrace{\frac{\partial f}{\partial z}}_{u} \underbrace{\frac{\partial^2 f}{\partial t \partial z} dz}_{dv}$$

integration by parts

$$= \int \mu \left(\frac{\partial f}{\partial t}\right)\frac{\partial^2 f}{\partial t^2} dz + T \frac{\partial f}{\partial z}\frac{\partial f}{\partial t}\Big|_a^b - \int_a^b T \frac{\partial^2 f}{\partial z^2}\frac{\partial f}{\partial t} dz$$

$$= \int_a^b \mu \left\{ \frac{\partial^2 f}{\partial t^2} - \frac{T}{\mu}\frac{\partial^2 f}{\partial z^2}\right\}\left(\frac{\partial f}{\partial t}\right) dz - T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)\Big|_{z=a} + T\left(\frac{\partial f}{\partial t}\right)\left(\frac{\partial f}{\partial z}\right)\Big|_{z=b}.$$

Now for waves on a string $\frac{\mu}{T} = \frac{1}{v^2}$ and the term in braces is zero. ∎

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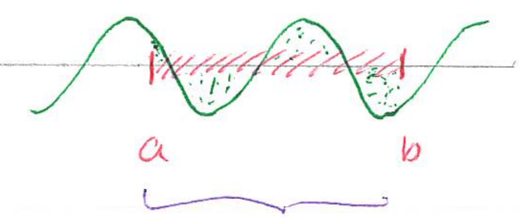
Thus the rate of energy change in $a \le z \le b$ is zero except for the contributions from the ends. We interpret these as energy flow rate. The

energy flow rate (in direction of increasing $z$) at any location is

$$\boxed{P = -T\left(\frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial z}}$$

and this is the power transmitted from left to right.

$P_a \rightarrow$ $\qquad$ $\underrightarrow{P_b} = -T\left(\frac{\partial f}{\partial t}\right)_b\left(\frac{\partial f}{\partial z}\right)_b$




$$E = \frac{1}{2}\int \left\{\mu\left(\frac{\partial f}{\partial t}\right)^2 + T\left(\frac{\partial f}{\partial z}\right)^2\right\} dz$$

energy in region