Fri: SPS

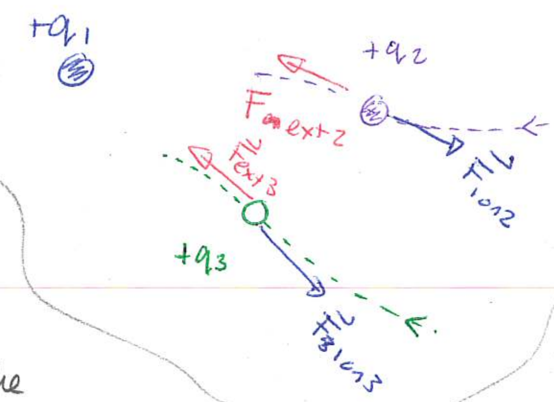
HW by 5pm

Tues: HW 4 by 5pm Read 6.1

Energy and fields in matter

Energy is required to assemble charges distributions. This includes energy required to create dipoles. In general the idea is

Point charges initially infinitely distant from each other and at rest $\longrightarrow$ Assemble charges so that they are eventually at rest



The process of assembling charges requires external forces to overcome the electrostatic forces. Let Wext be the work done by all external forces in the process of assembling the distribution. Let Welec be the work done by all electrostatic forces. Then

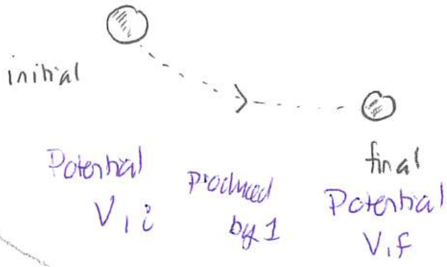
$$\underbrace{\Delta K}_{=0} = W_{ext} + W_{elec} \Rightarrow W_{ext} = -W_{elec}$$

We then define:

> The energy stored in the charge distribution is the work done by external forces to assemble the distribution starting with point charges infinitely far apart.

Quite generally the work done to assemble electrostatic charges is established via

Consider bringing q_2 into position near q_1. Then the work done by charge 1 on charge 2 is



$$W_{elec} = -q_2 \Delta V_1$$

The work done by 2 on 1 is

$$W_{elec} = -q_1 \Delta V_2 = -q_2 \Delta V_1$$

so for a collection of charges

$$W_{ext} = \frac{1}{2} \sum_{i \neq j} q_i \Delta V_j$$

or

$$W_{ext} = \frac{1}{2} \int_{\text{all space}} \rho(\vec{r}\,') V(\vec{r}\,') \, d\tau'$$

Using $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ and integration by parts gives

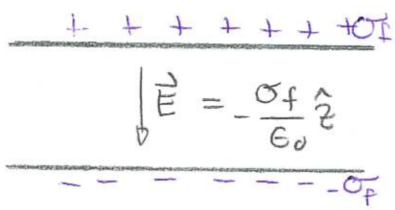
$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, d\tau$$

where $\vec{E}$ is the electric field produced by the assembled charges.

The considerations in the presence of matter that can be polarized are different

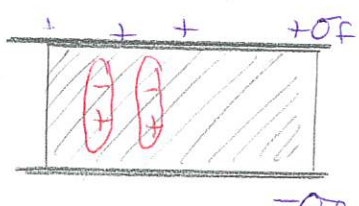
Consider two situations

Parallel plate capacitor no dielectric



$W = \frac{\epsilon_0}{2} \int \vec{E} \cdot \vec{E} \, d\tau$ gives work done to assemble all charges on plates

Parallel plate capacitor with dielectric



The work done to assemble the free charge will be different since:

* free charge must do some work to polarize the material
* bound charges will do some work on free charges

So we aim to determine the work done to assemble the free charge. We get

> The work required to assemble free charge in the presence of a linear dielectric is
>
> $$W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\,d\tau = \mathcal{U} \equiv \text{energy stored in the assembly of charges}$$
>
> where $\vec{E}$ is the field produced by all charges and $\vec{D}$ the total electric displacement.

Proof: Consider the process of adding free charge to a system

potential $V(\vec{r}\,')$ produced by all existing charge

Then the incremental work required is

$$\delta W = \delta q_f V(\vec{r}\,') = \int_{\text{all space}} \delta\rho_f d\tau' \; V(\vec{r}\,') = \int_{\text{all space}} \delta\rho_f V(\vec{r}\,') d\tau'$$

Then

$$\rho_f = \vec{\nabla}\cdot\vec{D} \Rightarrow \delta\rho_f = \vec{\nabla}\cdot(\delta\vec{D})$$

↳ incremental change in $\vec{D}$

giving

$$\delta W = \int_{\text{all space}} \left[\vec{\nabla}\cdot\delta\vec{D}\right] V(\vec{r}\,')d\tau'$$

Then a general vector identity is:

$$\vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} + f\,\vec{\nabla}\cdot\vec{A} \Rightarrow f\,\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(f\vec{A}) - \vec{\nabla}f\cdot\vec{A}$$

Here $\vec{A} = \delta\vec{D}$ and $f = V(\vec{r}\,')$ gives:

$$\int_{\text{all space}} (\vec{\nabla}\cdot\delta\vec{D})\, V(\vec{r}\,')\, d\tau' = \int_{\text{all space}} \vec{\nabla}\cdot(\delta\vec{D}\, V(\vec{r}\,'))\, d\tau' - \int \vec{\nabla} V(\vec{r}\,')\cdot\delta\vec{D}\, d\tau'$$

$$= \oint_{\substack{\text{infinite}\\ \text{boundary}}} V(\vec{r}\,')\,\delta\vec{D}\cdot d\vec{a} + \int_{\text{all space}} \vec{E}\cdot\delta\vec{D}\, d\tau'$$

where we have used $\vec{E} = -\vec{\nabla}V$. The integral over the surface at infinity gives zero since $V \to 0$ at infinity. Thus.

$$\delta W = \int_{\text{all space}} \delta\vec{D}\cdot\vec{E}\, d\tau$$

This gives the work done to add charge δQ where $\vec{E}$ is the field produced by all existing charge. Since this electric field is partly ~~determined by $\vec{D}$ the integral cannot be simplified further in general.~~

For a linear dielectric material,

$$\vec{D} = \epsilon\vec{E}$$

and

$$\delta\vec{D}\cdot\vec{E} = \epsilon(\delta\vec{E})\cdot\vec{E} = \frac{\epsilon}{2}\,\delta(\vec{E}\cdot\vec{E}) = \frac{\cancel{\epsilon}}{2}\,\delta\!\left(\frac{\vec{D}\cdot\vec{E}}{\cancel{\epsilon}}\right)$$

Thus

$$\delta W = \frac{1}{2}\int_{\text{all space}} \delta(\vec{D}\cdot\vec{E})\, d\tau = \delta\left[\frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau\right]$$

$$\Rightarrow \quad W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau \qquad \blacksquare$$

1 Energy stored in a parallel plate capacitor with dielectric

A capacitor consists of two parallel plates, each with area A and separated by distance d, which is much smaller than the dimensions of either plate. We aim to compare the energy stored in the capacitor with and without a linear dielectric. In both cases we will assume that the plates are charged uniformly with equal and opposite free charges of magnitude Q_f.

a) Without calculating any fields, will the energy stored when the dielectric is present be less than, more than or the same as when the dielectric is absent? Explain your answer.

b) Determine the energy stored in terms of Q_f, A, d and the dielectric constant.

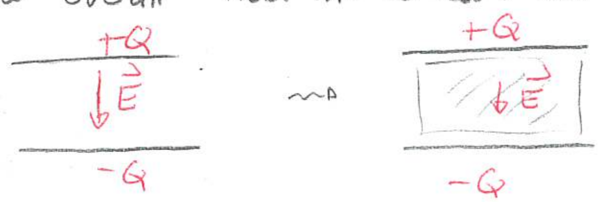
c) By what fraction is the energy different to that if the dielectric were absent?

d) Determine the energy stored in the capacitor in terms of the capacitance

$$C = \frac{\epsilon A}{d}$$

and Q_f.

Answer a). The overall field will be less. Thus the energy stored in the field will be less with the dielectric



b) The energy stored is:

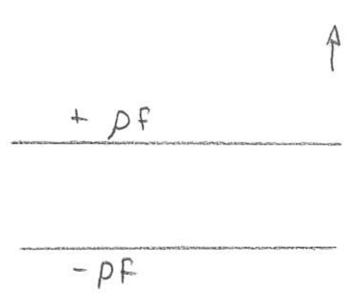
$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau$$

and we need $\vec{D}$ and $\vec{E}$. This will only be non-zero between the plates. The strategy is

Given Q_F $\longrightarrow$ Find $\vec{D}$ via $\vec{\nabla} \cdot \vec{D} = \rho_f$ OR $\oint \vec{D} \cdot d\vec{a} = q_{f\,enc}$ $\longrightarrow$ Find $\vec{E}$ via $\vec{E} = \frac{1}{\epsilon} \vec{D}$

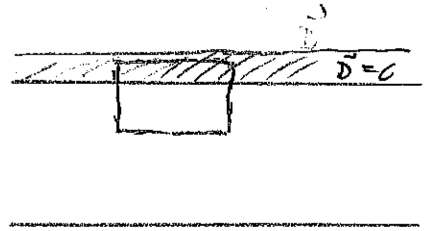
To find $\vec{D}$ we note that symmetry implies

$$\vec{D} = D_z(z) \hat{z}$$



We use a pill box as a Gaussian surface

Then on the top surface, with area a



$$\int \vec{D}\cdot d\vec{a} = \int_{top} \vec{D}\cdot \hat{z}\, da = 0$$

since $\vec{D}=0$ inside the conducting plates.

On the bottom surface $d\vec{a} = -da\,\hat{z}$

$$\int \vec{D}\cdot d\vec{a} = -\int_{bottom} D_z(z)\, da = -D_z(z)\, a$$

On the side surfaces $d\vec{a}$ is perpendicular to $\vec{D}$. Thus

$$\int_{side} \vec{D}\cdot d\vec{a} = 0$$

Adding all gives.

$$\oint \vec{D}\cdot d\vec{a} = q_{free\ enc} \Rightarrow -D_z(z)\, a = \sigma_f\, a$$

$$\Rightarrow D_z(z) = -\sigma_f$$

Thus

$$\vec{D} = \begin{cases} 0 & \text{outside} \\ -\sigma_f\,\hat{z} & \text{inside.} \end{cases}$$

Then

$$\vec{E} = \begin{cases} \sigma & \text{outside} \\ -\dfrac{\sigma_f}{\epsilon}\,\hat{z} & \text{inside.} \end{cases}$$

So the energy stored is

$$U = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau = \frac{\sigma_f^2}{2\epsilon}\underbrace{\int_{inside} d\tau}_{\text{volume between plates } = dA}$$

$$U = \frac{\sigma_f^2\, dA}{2\epsilon}$$

Then $Q_F = \sigma_F A \Rightarrow \sigma_F = Q_f/A$

Thus

$$\boxed{U = \frac{Q_F^2 d}{2\epsilon A}}$$

c) without the dielectric

$$U_{without} = \frac{Q_F^2 d}{2\epsilon_0 A}$$

with the dielectric

$$U_{with} = \frac{Q_f^2 d}{2\epsilon A}$$

But $\epsilon = \epsilon_0 \epsilon_r \Rightarrow$ $$U_{with} = \frac{Q_f^2 d}{2\epsilon_0 \epsilon_r A} = \frac{1}{\epsilon_r} U_{without}.$$

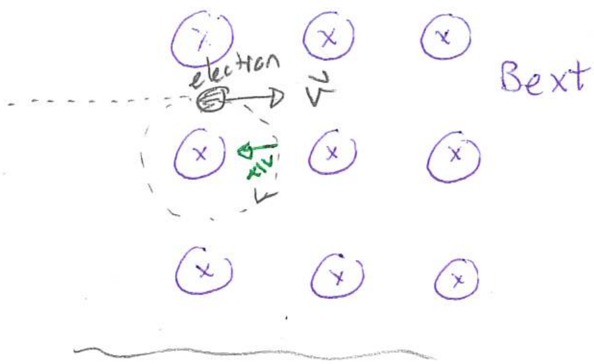
Since $\epsilon_r > 1$ the energy with is $1/\epsilon_r$ of energy without

d) $$U_{with} = \frac{Q_F^2}{2}\left(\frac{d}{\epsilon A}\right) = \frac{1}{2}\frac{Q_F^2}{C}$$

Magnetism in matter

In the same way that matter can respond to external electric fields, it can also respond to external magnetic fields. This can have an effect on the total magnetic field.

Consider, for example an electron in a uniform magnetic field. The field exerts a force on the electron. This is determined via



$$\vec{F} = q\,\vec{v} \times \vec{B}_{ext}$$

That causes the electron to orbit. The orbiting electron can be regarded as a current loop

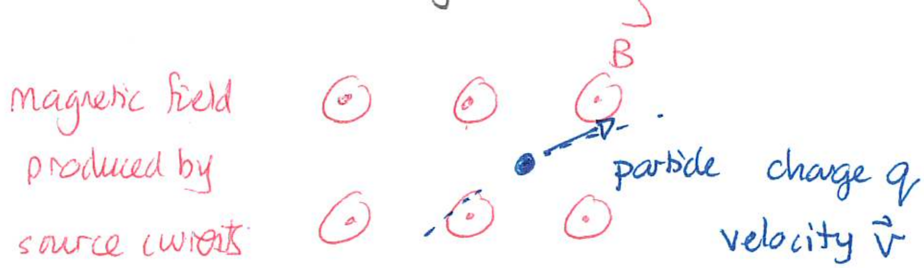
This current produces a magnetic field opposite to the external field. This is a diamagnetic effect and is observed in situations like NMR.

The basic model we use for magnetism in materials is to regard the material as containing magnetic dipoles. So we will need

* a description of an isolated magnetic dipole
* a " " the field produced by a magnetic dipole
* a " " the force and torque produced by an external magnetic field on a magnetic dipole.

Magnetic fields and forces

The basic rule for magnetic force considers a particle in a field produced by other sources.

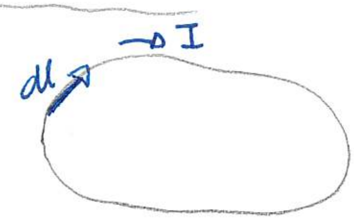


Then the force exerted by the magnetic field on the particle is

$$\vec{F} = q(\vec{v} \times \vec{B})$$

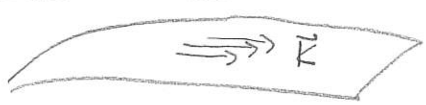
These can be generalized to forces exerted on currents:

Linear current



$$\vec{F} = \int_{\text{loop}} I \, d\vec{l} \times \vec{B}$$

Surface current



$$\vec{K} = \sigma \vec{v}$$

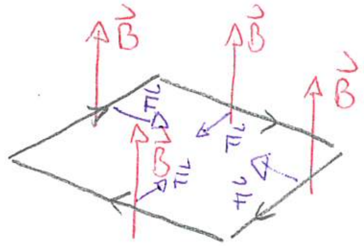
$$\vec{F} = \int_{\text{surface}} \vec{K} \times \vec{B} \, da$$

Volume current

$$\vec{J} = \rho \vec{v}$$

$$\vec{F} = \int_{\text{volume}} \vec{J} \times \vec{B} \, d\tau$$

We aim to apply this to loops of current such as that illustrated: where a square loop is in a uniform magnetic field. We can see that



* the forces on opposite sides cancel

$$\Rightarrow \vec{F}_{net} = 0$$

* the net torque about the center is not zero.

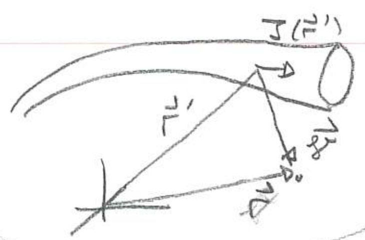
We want to describe the torque in terms of the current configuration of the loop and this is described by magnetic dipoles.

Magnetic dipoles

A magnetic field can be calculated from a magnetic vector potential:

Magnetic vector potential

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$



Magnetic field

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

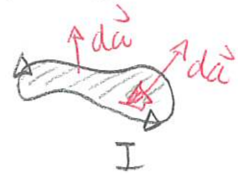
Multipole expansion

$$\vec{A}(\vec{r}) = \vec{A}_{dipole}(\vec{r}) + \ldots \text{smaller terms}$$

where

$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^2}$$

Magnetic dipole moment



$$\vec{m} = \int I\,d\vec{a}$$

surface enclosing current loop

For a point magnetic dipole $\vec{m}$ does not depend on $\vec{r}$ and vector calculus gives:

> The magnetic field produced by a magnetic dipole $\vec{m}$ at location $\vec{r}$ is:
>
> $$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\left\{3(\vec{m}\cdot\hat{r})\,\hat{r} - \vec{m}\right\}$$