Fri: HW 1 by 5pm

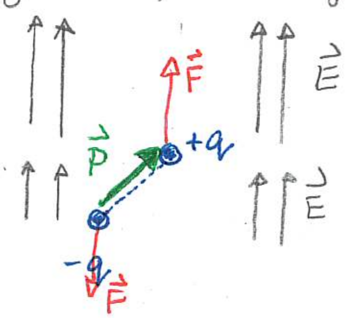
Tues: HW 2 by 5pm

Read:

Electric dipoles in external fields

We will consider models of materials which consist of electric dipoles. The response of these to external electric fields will be determined by the response of a dipole to an external field. Consider a dipole consisting of two point charges placed in an external field. The field exerts a force on each point charge, given by

$$\vec{F} = q\vec{E}$$



When applied to both charges we get:

1) The net force on the dipole
   - is zero if the external electric field is uniform
   - could be non-zero if the external electric field is non-uniform and has some type of gradient.

2) The net torque on the dipole
   - depends on the alignment of the dipole with the field
   - could be non-zero even if the field is uniform.

The derivation of these comes from basic electrostatics.

The general results are

> Consider a charge distribution whose net charge is zero. Suppose the distribution is placed in an external electric field $\vec{E}_{ext}(\vec{r})$. In the dipole approximation, the net force on the dipole is
>
> $$\vec{F} = (\vec{p}\cdot\vec{\nabla})\,\vec{E}_{ext}$$
>
> and the net torque in a uniform field is
>
> $$\vec{\tau} = \vec{p}\times\vec{E}_{ext}$$
>
> where
>
> $$(\vec{p}\cdot\vec{\nabla})\,\vec{A} = p_x\frac{\partial\vec{A}}{\partial x} + p_y\frac{\partial\vec{A}}{\partial y} + p_z\frac{\partial\vec{A}}{\partial z}$$

Proof: (Net force)

The proof requires a lemma: For any field $\vec{E}_{ext}(\vec{r})$,

$$\vec{E}_{ext}(\vec{r}\,') = \vec{E}_{ext}(\vec{r}) + \left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r}) + \ldots$$

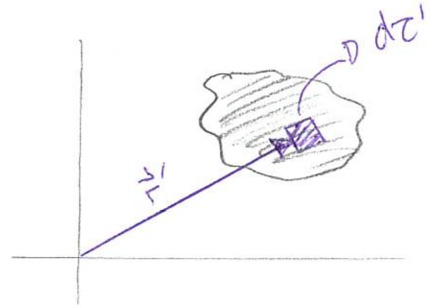
where $\vec{\nabla}$ indicates differentiation w.r.t. $\vec{r}$ (unprimed). To show this consider a scalar function $g(\vec{r}\,') = g(x',y',z')$. Then

$$g(x',y',z') = g(x,y,z) + (x'-x)\left.\frac{\partial g}{\partial x}\right|_{x,y,z} + (y'-y)\left.\frac{\partial g}{\partial y}\right|_{x,y,z} + (z'-z)\left.\frac{\partial g}{\partial z}\right|_{x,y,z} + \ldots$$

$$g(\vec{r}\,') = g(\vec{r}) + (\vec{r}\,'-\vec{r})\cdot(\vec{\nabla}g)_{\text{evaluated at }\vec{r}}$$

This would apply separately to each component of $\vec{E}_{ext}(\vec{r}\,')$ and that proves the lemma.

Now the force on the dipole is determined via decomposition.



$$\vec{F} = \int \rho(\vec{r}\,')d\tau' \, \vec{E}_{ext}(\vec{r}\,')$$

$$= \int_{\text{all space}} \rho(\vec{r}\,')\vec{E}_{ext}(\vec{r}\,')\,d\tau'$$

$$= \int_{\text{all space}} \rho(\vec{r}\,')\left[E_{ext}(\vec{r}) + (\vec{r}\,'-\vec{r})\cdot\vec{\nabla}(\vec{E}_{ext})\right]d\tau' + \dots$$

$$= \int_{\text{all space}} \rho(\vec{r}\,')\vec{E}_{ext}(\vec{r})\,d\tau' + \int \rho(\vec{r}\,')\left[(\vec{r}\,'-\vec{r})\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r})\,d\tau' + \dots$$

Since $\vec{E}_{ext}(\vec{r})$ in these does not depend on the primed integration variable,

$$\vec{F} = \left[\int \rho(\vec{r}\,')d\tau'\right]\vec{E}_{ext}(\vec{r}) + \left[\int \rho(\vec{r}\,')\,\vec{r}\,'\cdot\vec{\nabla}\,d\tau'\right]\vec{E}_{ext}(\vec{r})$$

$$- \left[\int \rho(\vec{r}\,')d\tau'\,\vec{r}\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r}) + \dots$$

Then $\int \rho(\vec{r}\,')d\tau'$ is the total charge. If this is zero then

$$\vec{F} = \left[\underbrace{\int \rho(\vec{r}\,')\vec{r}\,'d\tau'}_{\text{dipole moment } \vec{p}}\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r}) + \dots$$

$$\vec{F} = \left[\vec{p}\cdot\vec{\nabla}\right]\vec{E}_{ext}(\vec{r}) + \dots$$

Ignoring the smaller higher order terms gives the force result.

Proof (Net torque)

The force on the distribution at location $\vec{r}'$ is

$$d\vec{F} = \rho(\vec{r}')\, d\tau'\, \vec{E}_{ext}(\vec{r}')$$

Then the torque on this is

$$d\vec{\tau} = \vec{r}' \times d\vec{F}$$

$$= \vec{r}' \times \rho(\vec{r}')\, d\tau'\, \vec{E}_{ext}(\vec{r}')$$

and the net torque is:

$$\vec{\tau} = \int \rho(\vec{r}')\, \vec{r}' \times \vec{E}_{ext}(\vec{r}')\, d\tau'$$

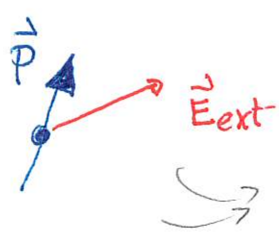
$$= \int \rho(\vec{r}')\, \vec{r}'\, d\tau' \times \vec{E}_{ext}(\vec{r}) + \int \rho(\vec{r}')\, \vec{r}' \times (\vec{r}' - \vec{r}) \cdot \vec{\nabla}\, \vec{E}_{ext}(\vec{r})$$

If the field is uniform, then the second term is zero and

$$\vec{\tau} = \underbrace{\int \rho(\vec{r}')\, \vec{r}'\, d\tau'}_{\vec{p}} \times E_{ext}(\vec{r})$$

This gives the result. ∎

This applies to a point dipole and we need only use the electric field at the dipole location.



these two vectors give torque

Potential energy of a dipole in an external field

In general there exists a potential energy U associated with a force $\vec{F}$ if

$$\vec{F} = -\vec{\nabla} U.$$

Then we know that

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E}_{ext}$$

But a vector identity is:

$$\vec{\nabla}(\vec{p} \cdot \vec{E}) = \vec{p} \times (\vec{\nabla} \times \vec{E}) + \vec{E} \times (\vec{\nabla} \times \vec{p}) + (\vec{p} \cdot \vec{\nabla}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{p}$$

In this case $\vec{p}$ does not depend on $\vec{r}$, so $\vec{\nabla} \times \vec{p} = 0$ and the last term is zero. So

$$\vec{\nabla}(\vec{p} \cdot \vec{E}) = (\vec{p} \cdot \vec{\nabla}) \vec{E} + \vec{p} \times (\vec{\nabla} \times \vec{E})$$

For electrostatic fields $\vec{\nabla} \times \vec{E} = 0$. This gives that, for electrostatic external fields

$$\boxed{\vec{F} = \vec{\nabla}(\vec{p} \cdot \vec{E}_{ext})}$$

We immediately see that

> For a dipole in an external electrostatic field $\vec{E}_{ext}$, the potential energy associated with the interaction is
>
> $$U = -\vec{p} \cdot \vec{E}_{ext}.$$

1 Dipole in external electric fields

A sphere with radius R has a charge distribution given in spherical coordinates by

$$\rho(\mathbf{r}') = \frac{8\alpha}{\pi^2 R^3}\ \sin\phi'$$

where $\alpha > 0$ has units of Coulombs.

a) What is the net charge of the sphere?

b) What is the direction of the sphere's electric dipole moment?

c) The sphere is placed in the external electric field

$$\mathbf{E} = \beta z\ \hat{\mathbf{z}}$$

where $\beta > 0$ is a constant with units of N/Cm. Determine the net force and torque on the sphere (in the dipole approximation).

d) Provide the simplest field that would result in a non-zero net force on the on the sphere (in the dipole approximation).

e) The sphere is placed in the external electric field

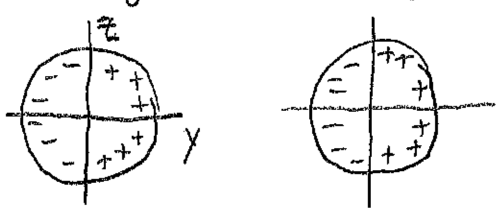
$$\mathbf{E} = \beta z\ \hat{\mathbf{z}}$$

where $\beta > 0$ is a constant with units of N/Cm. Which way would the sphere align so as to yield the lowest potential energy?

Answer: a) $Q = \int \rho(\vec{r}')d\tau' = \int_0^R dr' \int_0^\pi d\theta' \int_0^{2\pi} d\phi'\ r'^2 \sin\theta'\ \frac{8\alpha}{\pi^2 R^3} \sin\phi'$

The integral w.r.t. ϕ' gives 0. So $Q=0$

b)



It appears to be $\longrightarrow$ $\boxed{\vec{p} = p\,\hat{y}}$

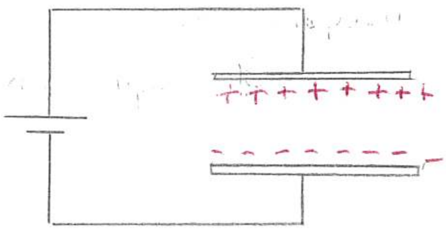
c) $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} = p\frac{\partial}{\partial y}\vec{E} = 0$ $\quad \boxed{\vec{F} = 0}$

d) Anything that gives $\frac{\partial \vec{E}}{\partial y} \neq 0$ will work. So $\boxed{\vec{E} = \beta y\,\hat{y}}$ is a possibility

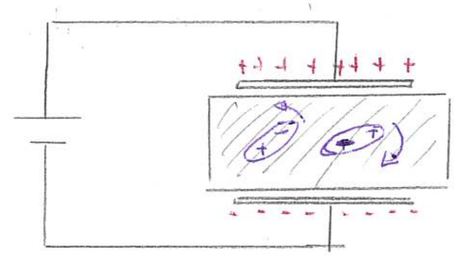
e) The energy is $U = -\vec{p}\cdot\vec{E} = -\beta z p_z$. So we need p_z to be as large as possible. The sphere will align with $\vec{p}$ along the $\hat{z}$-axis

Material in external electric fields: polarization

Now consider a material that contains electric dipoles and which is placed in an external electric field, for example produced by a parallel plate capacitor.



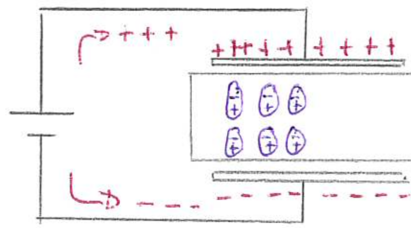
insert → material



material

dipoles realign

earlier



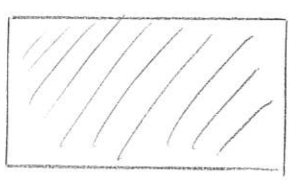
dipoles aligned
↳ more charge appears on plates.

DEMO: PhET Capacitor Lab → Dielectric Tab

1) Connect plates with dielectric out
   - keep battery connected and slide dielectric in

2) Connect plates with dielectric out + charge
   - disconnect plates
   - slide dielectric in/out
   - observe polarization
   - observe voltage

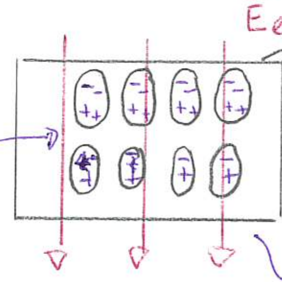
The fundamental way to describe this considers charges inside the material. These are called bound charges since they are constrained to the material. We can view this as follows. The capacitor produces an external electric field, Eext that polarizes the material



no field - zero charge density

$\vec{E}_{ext}$

interior has a positive bound charge.



surface has a negative bound charge

↳ surface has a positive bound charge

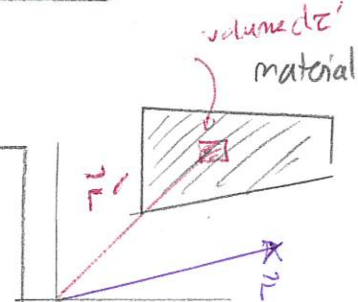
We will describe these bound charges by:

$$\boxed{\begin{aligned} \rho_b(\vec{r}\,') &= \text{volume bound charge density inside the material as a result of polarization} \\ \sigma_b(\vec{r}\,') &= \text{surface bound charge density on the surface of the material as a result of polarization.} \end{aligned}}$$

Regardless of how these arise they produce an electrostatic potential via the usual rules. Denote this potential

$$\boxed{V_p = \text{potential produced by bound charges}}$$

Thus regardless of how the polarization arises.



$$\boxed{\begin{gathered} V_p(\vec{r}) = \frac{1}{4\pi\epsilon_0}\underbrace{\int \frac{\rho_b(\vec{r}\,')}{\mathscr{r}}\,d\tau'}_{\substack{\text{region} \\ \text{(material)}}} + \frac{1}{4\pi\epsilon_0}\underbrace{\int \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}}\,da'}_{\substack{\text{surface} \\ \text{(material)}}} \\ \text{where} \quad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,' \end{gathered}}$$

Furthermore if the material is electrically neutral then the total charge is zero

Thus

$$\int_{\text{volume}} \rho_b(\vec{r}\,')\,d\tau' + \int \sigma_b(\vec{r}\,')\,da' = 0$$

We now look for ways to determine the bound charge densities. An intuitive way (not necessarily always correct) is to focus on the electric dipole moment within infinitesimally small regions. We then introduce the concept

polarization = dipole moment per unit volume

Specifically

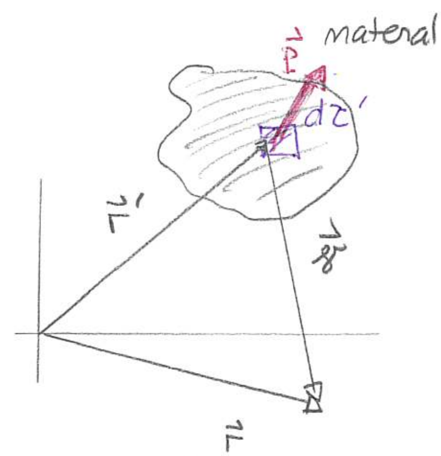
The polarization of a material is a vector $\vec{P}(\vec{r}')$ such that the electric dipole moment in region of volume $d\tau'$ at $\vec{r}'$ is

$$d\vec{p} = \vec{P}(\vec{r}')d\tau'$$

Then the polarization has units of C/m^2.

Potential from polarization

Given a material with polarization $\vec{P}(\vec{r}')$ the potential from the shaded segment is:



$$dV_p = \frac{1}{4\pi\epsilon_0}\frac{d\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Integration yields

$$V_p = \frac{1}{4\pi\epsilon_0}\int_{\text{material}}\frac{\vec{P}(\vec{r}')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

There now follow vector calculus operations. First

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$$

where differentiation is with respect to primed $\vec{\nabla}' = \frac{\partial}{\partial x'}\hat{x} + \frac{\partial}{\partial y'}\hat{y} + \frac{\partial}{\partial z'}\hat{z}$.

Second

$$\vec{\nabla}'\cdot(f\vec{A}) = f\,\vec{\nabla}'\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}'f$$

$$\Rightarrow \quad \vec{A}\cdot\vec{\nabla}'f = \vec{\nabla}'\cdot(f\vec{A}) - f(\vec{\nabla}'\cdot\vec{A})$$

So

$$V_P = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int_{\text{material}} \vec{\nabla}\cdot\left(\vec{P}\,\frac{1}{\mathscr{r}}\right) d\tau' - \frac{1}{4\pi\epsilon_0}\int_{\text{material}} \frac{1}{\mathscr{r}}\left(\vec{\nabla}'\cdot\vec{P}\right) d\tau'$$

$\Downarrow$ divergence theorem

$$= \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{surface}\\\text{material}}} \frac{\vec{P}(\vec{r}\,')}{\mathscr{r}}\cdot d\vec{a}\,' - \frac{1}{4\pi\epsilon_0}\int_{\substack{\text{volume}\\\text{material}}} \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

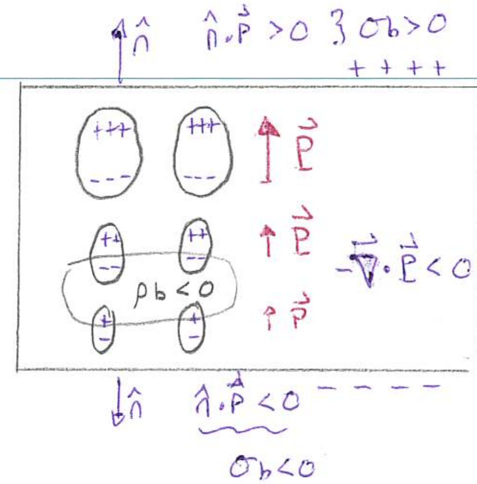
$$= \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\vec{P}(\vec{r}\,')\cdot\hat{n}'}{\mathscr{r}}\, da' - \frac{1}{4\pi\epsilon_0}\int_{\text{volume}} \frac{\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')}{\mathscr{r}}\, d\tau'$$

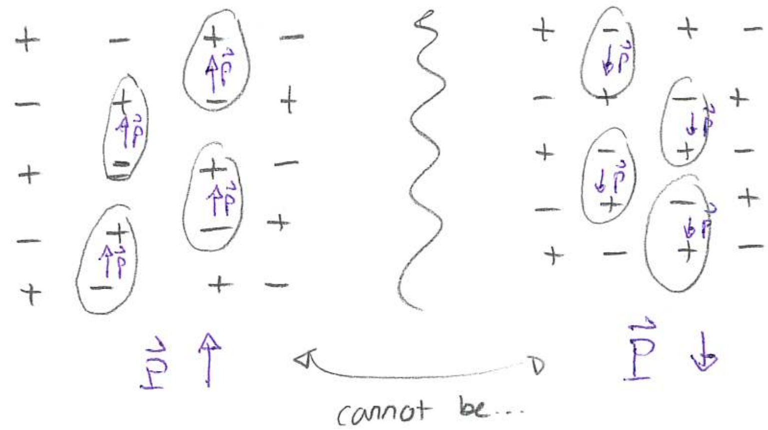
$$= \frac{1}{4\pi\epsilon_0}\int_{\text{volume}} \frac{\rho_b(\vec{r}\,')}{\mathscr{r}}\, d\tau' + \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\sigma_b(\vec{r}\,')}{\mathscr{r}}\, da'$$

This will match the general potential calculation from bound charges provided that

> The bound surface charge density is
>
> $$\sigma_b = \vec{P}(\vec{r}\,')\cdot\hat{n}'$$
>
> where $\hat{n}'$ is an outward normal surface vector.
>
> The bound volume charge density is
>
> $$\rho_b = -\vec{\nabla}'\cdot\vec{P}(\vec{r}\,')$$

In the context of electric dipoles within the material, we see that makes some intuitive sense. There are some deeper conceptual difficulties that limit the polarization picture. For example, in an ionic crystal the polarization is ambiguous.





These difficulties are avoided by the following.

> The polarization is a vector quantity $\vec{P}(\vec{r}')$ such that the bound volume charge density is
>
> $$\rho_b = -\vec{\nabla}\cdot\vec{P}$$
>
> and the bound surface charge density is
>
> $$\sigma_b = \vec{P}\cdot\hat{n}$$

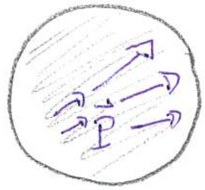
In this picture there are many possible polarizations that give the same bound charge densities. Note that if they are constructed from this polarization vector, then: the total charge is:

$$\int_{\text{vol}} \rho_b \, d\tau' + \int_{\text{surface}} \sigma_b \, da' = -\int_{\text{vol}} \vec{\nabla}\cdot\vec{P}\, d\tau' + \underbrace{\int_{\text{surface}} \hat{P}\cdot d\vec{a}}_{=\int_{\text{vol}} \vec{\nabla}\cdot\vec{P}\, d\tau'} = 0$$

This arrangement automatically satisfies the requirement that the total charge is zero.

We now have a scheme for describing the electrostatic potential and field produced by the bound charges (or equivalently the polarization).

Describe polarization

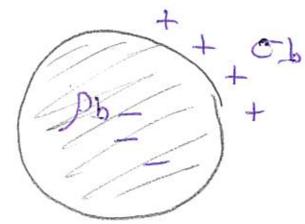


Determine bound surface charge density

$\sigma_b = \vec{P} \cdot \hat{n}$

and bound volume charge density

$\rho_b = -\vec{\nabla} \cdot \vec{P}$



Compute potential using standard methods for bound charges

Compute electric fields using standard methods for bound charges

Coulomb's Law, Gauss' Law

2 Bound charge distributions: radial polarization

A sphere of radius R has polarization

$$\mathbf{P} = \alpha r^2 \hat{\mathbf{r}}$$

where $\alpha > 0$ is a constant with units $\mathrm{C/m^4}$.

a) Draw a qualitative sketch of the dipole distribution within the sphere.

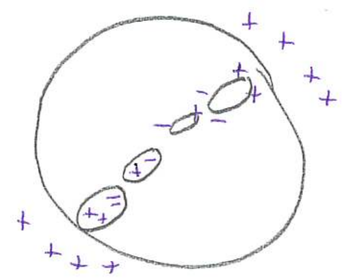
b) Do you expect that either the bound surface or bound volume charge densities will be zero? If not what do you expect their signs will be?

c) Determine the bound surface and volume charge densities.

d) Determine the total charge on the sphere.

e) Determine the electric field at all locations.

Answer: a)



b) $\sigma_b > 0$

$\rho_b < 0$

c) For the surface $\hat{n} = \hat{r}$ and $\sigma_b = \vec{P} \cdot \hat{n} \Rightarrow \sigma_b = \alpha r^2 > 0$

For the volume

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) + \ldots \cancel{\frac{\partial}{\partial \theta}(\sin\theta P_\theta)} + \ldots \cancel{\frac{\partial P_\phi}{\partial \phi}} \right\}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\alpha r^4 = -\frac{1}{r^2}\alpha 4 r^3$$

$$\Rightarrow \rho_b = -4\alpha r < 0$$

d) On the surface

$$\int \sigma_b \, da' = \int \alpha R^2 \, da'$$

$\left.\begin{array}{l} r' = R \\ 0 \le \theta' \le \pi \\ 0 \le \phi' \le 2\pi \end{array}\right\}$ $da' = r'^2 \sin\theta' d\theta' d\phi' = R^2 \sin\theta' d\theta' d\phi'$

surface charge

$$\int_0^\pi d\theta' \int_0^{2\pi} d\phi' \, R^2 \sin\theta' \, \sigma_b = 4\pi R^2 \alpha R^2$$

$$= 4\pi \alpha R^4.$$

In the volume

$$\int \rho_b \, d\tau' = \int_0^R dr' \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \, r'^2 \sin\theta' \, (-4\alpha r')$$

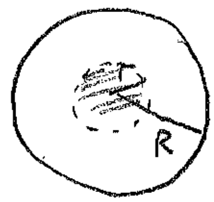
$$= -4\alpha \underbrace{\int_0^R r'^3 dr'}_{R^4/4} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^{\pi} d\theta' \sin\theta'}_{2} = -4\pi\alpha R^4$$

The surface and volume charges add to zero.

e) Use Gauss' Law. * By symmetry $\vec{E}(\vec{r}) = E_r(r)\hat{r}$

* use a Gaussian sphere with radius r
* on this sphere

$$\left.\begin{array}{c} r = r \\ 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{array}\right\} \quad d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$



Thus

$$\vec{E} \cdot d\vec{a} = E_r(r) \, r^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \oint_{\text{surface}} \vec{E} \cdot d\vec{a} = E_r(r) r^2 \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi r^2 E_r(r)$$

Gauss' Law gives

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_0} \Rightarrow E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

If $r \geq R$ then $q_{enc} = 0$

If $r < R$ then $q_{enc} = \int \rho_b \, d\tau' = \int_0^r dr' \int_0^{\pi} d\theta' \int_0^{2\pi} d\phi' \, r'^2 \sin\theta' \, (-4\alpha r')$

$$= -4\alpha \underbrace{\int_0^r r'^3 dr'}_{r^4/4} \underbrace{\int_0^{\pi} \sin\theta' d\theta'}_{2} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi}$$

$$\Rightarrow q_{enc} = -4\pi\alpha r^4$$

Thus

$$\vec{E}(\vec{r}) = \begin{cases} -\dfrac{\alpha r^2}{\epsilon_0} \hat{r} & \text{if } r < R \\ 0 & \text{if } r > R \end{cases}$$