Electromagnetic Theory II: Class Exam II

8 April 2021

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Instructions

- There are 6 questions on 7 pages.
- Show your reasoning and calculations and always explain your answers.

Physical constants and useful formulae

Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$
Charge of an electron	$e = -1.60 \times 10^{-19}\,\mathrm{C}$
Speed of light	$c = 3.0 \times 10^{8}\,\mathrm{m/s}$

$$\int \sin(ax)\sin(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \cos(ax)\cos(bx)\,\mathrm{d}x = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \quad \text{if } a \neq b$$

$$\int \sin(ax)\cos(ax)\,\mathrm{d}x = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\,\mathrm{d}x = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

$$\int x\sin^2(ax)\,\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$$

$$\int x^2\sin^2(ax)\,\mathrm{d}x = \frac{x^3}{6} - \frac{x^2}{4a}\sin(2ax) - \frac{x}{4a^2}\cos(2ax) + \frac{1}{8a^3}\sin(2ax)$$

Question 1

An electromagnetic wave has the complex representation

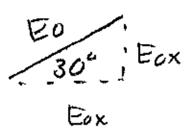
$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 \, e^{i(kz-\omega t)}$$

where $\tilde{\mathbf{E}}_0$ is independent of position and time.

a) Determine an expression for $\tilde{\mathbf{E}}_0$ such that the wave is linearly polarized along the line in the xy plane that is $30°$ above the x-axis.

Linearly pol $\quad \tilde{\vec{E}}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} \quad \Rightarrow \vec{E} = (E_{0x}\hat{x} + E_{0y}\hat{y}) e^{i(kz-\omega t)}$

$$\vec{E} = (E_{0x}\hat{x} + E_{0y}\hat{y}) \cos(kz-\omega t)$$



$$= (E_0 \cos 30° \, \hat{x} + E_0 \sin 30° \, \hat{y}) \cos(kz-\omega t)$$

$$\tilde{\vec{E}}_0 = E_0 (\cos 30° \, \hat{x} + \sin 30° \, \hat{y})$$

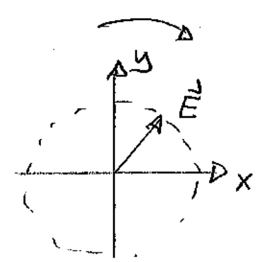
$$= \frac{E_0}{2}\left[\sqrt{3}\,\hat{x} + \hat{y}\right]$$

b) Suppose that $\tilde{\mathbf{E}}_0 = E_0\left(\hat{\mathbf{x}} + e^{-i\pi/2}\hat{\mathbf{y}}\right)$ where $E_0 > 0$. Describe the polarization of this wave. Explain your answer.

$$\tilde{\vec{E}} = E_0 e^{i(kz-\omega t)} \hat{x} + E_0 e^{i(kz-\omega t - \pi/2)} \hat{y}$$

Real $\rightarrow$ $\vec{E} = E_0 \cos(kz-\omega t)\hat{x} + E_0 \underbrace{\cos(kz-\omega t - \pi/2)}_{\sin(kz-\omega t)}\hat{y}$

$$= E_0 \cos(kz-\omega t)\hat{x} + E_0 \sin(kz-\omega t)\hat{y}$$



Circularly polarized. Viewed as illustrated rotates c.w

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At $z=0$

$$\vec{E} = E_0 \cos\omega t \, \hat{x} - E_0 \sin\omega t \, \hat{y}$$

Question 2

Consider the following candidates for electromagnetic waves in a vacuum. In each case $k > 0$ and $\omega > 0$.

$$
\begin{array}{ll}
\text{Case A:} & \mathbf{E} = A\hat{\mathbf{y}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = A\hat{\mathbf{z}}\, e^{i(kx-\omega t)} \\
\text{Case B:} & \mathbf{E} = A\hat{\mathbf{y}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = -A\hat{\mathbf{z}}\, e^{i(kx-\omega t)} \\
\text{Case C:} & \mathbf{E} = A\hat{\mathbf{y}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = \frac{A}{c}\hat{\mathbf{z}}\, e^{i(kx-\omega t)} \\
\text{Case D:} & \mathbf{E} = A\hat{\mathbf{y}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = -\frac{A}{c}\hat{\mathbf{z}}\, e^{i(kx-\omega t)} \\
\text{Case E:} & \mathbf{E} = A\hat{\mathbf{x}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = \frac{A}{c}\hat{\mathbf{y}}\, e^{i(kx-\omega t)} \\
\text{Case F:} & \mathbf{E} = A\hat{\mathbf{x}}\, e^{i(kx-\omega t)} \text{ and } \mathbf{B} = -\frac{A}{c}\hat{\mathbf{y}}\, e^{i(kx-\omega t)}
\end{array}
$$

Which of these represent possible electromagnetic waves in a vacuum? Explain your answer.

Each has direction along x $\Rightarrow$ $\vec{E}, \vec{B}$ in y,z plane $\Rightarrow$ Not E,F.

$\vec{B} = \frac{1}{c}(\hat{k} \times \vec{E})$ $\Rightarrow$ magnitude B = $\frac{1}{c}$ magnitude E $\Rightarrow$ Not A,B

for C,D $\quad \hat{B} = \frac{1}{c}(\hat{x} \times \vec{E}) = \frac{1}{c}\, \hat{x} \times A\hat{y}\, e^{i(kx-\omega t)}$

$= \frac{1}{c} A \hat{z}\, e^{i(kx-\omega t)}$ $\Rightarrow$ only $\boxed{C}$

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Question 3

An electromagnetic wave is incident perpendicular to a surface that separates two media. The fraction of incident energy that is reflected is 16/25. Determine the ratio of the indices of refraction of the two media, assuming that the permeability is the same for the two materials (i.e. $\mu_1 = \mu_2 = \mu_0$). Assuming $n_2 < n_1$

$$R = \frac{I_R}{I_I} = \frac{16}{25}$$

$$I_R = \frac{1}{2} \epsilon_1 v_1 |E_{OR}|^2$$

$$I_T = \frac{1}{2} \epsilon_1 v_1 |E_{OI}|^2$$

$$R = \frac{|E_{OR}|^2}{|E_{OI}|^2}$$

$$E_{OR} = \left(\frac{1-\beta}{1+\beta}\right) E_{OI}$$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{v_1}{v_2}$$

$$R = \left(\frac{1-\frac{v_1}{v_2}}{1+\frac{v_1}{v_2}}\right)^2$$

$$= \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 \Rightarrow \frac{16}{25} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)^2 \Rightarrow \frac{v_2 - v_1}{v_2 + v_1} = \frac{4}{5}$$

$$\Rightarrow 5v_2 - 5v_1 = 4v_2 + 4v_1$$

$$\Rightarrow v_2 = 9v_1$$

Then $n = c/v \Rightarrow v = \frac{c}{n} \Rightarrow \frac{c}{n_2} = 9\frac{c}{n_1} \Rightarrow n_1 = 9n_2$

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Question 4

Consider a region in which there are no source charges or currents. Starting with Maxwell's equations, show that the electric field satisfies the wave equation and provide an expression for the wave speed in terms of electromagnetic constants.

$$\vec{\nabla} \cdot \vec{E} = \cancel{\rho/\epsilon_0} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \cancel{\mu_0 \vec{J}} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \overbrace{(\cancel{\vec{\nabla} \cdot \vec{E}})}^{0} - \nabla^2 \vec{E}$$

$$\Rightarrow \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\nabla^2 \vec{E}$$

$$\Rightarrow -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\nabla^2 \vec{E}$$

$$\Rightarrow -\frac{\partial}{\partial t}\left(\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right) = -\nabla^2 \vec{E}$$

$$\Rightarrow \nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$\Rightarrow$ of form

$$\nabla^2 f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$$

speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

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Question 5

Certain source charges and currents produce the scalar potential

$$V = \begin{cases} 0 & \text{if } z > d \\ kze^{-t/\tau} & \text{if } d > z > -d \\ 0 & \text{if } z < -d \end{cases}$$

where $k > 0$ is a constant with units of V/m and $\tau > 0$ is a constant with units of time. Throughout all space the vector potential is $\mathbf{A} = 0$.

a) Determine the electric and magnetic fields associated with these potentials.

$$\vec{E} = -\vec{\nabla} V - \cancel{\frac{\partial \vec{A}}{\partial t}}$$

$$= \begin{cases} -ke^{-t/\tau}\hat{z} & d > z > -d \\ 0 & \text{otherwise} \end{cases}$$

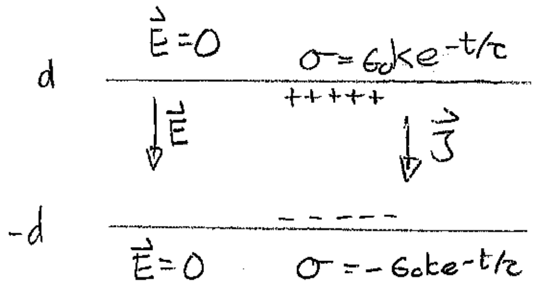
$$\vec{B} = \vec{\nabla} \times \vec{A} = 0$$

b) Determine the charge and current densities that produce these potentials. Sketch these and describe the physical situation that might give rise to such potentials and fields.

$$\cancel{\nabla^2 V} + \frac{\partial}{\partial t}\overset{0}{\cancel{(\vec{\nabla}\cdot\vec{A})}} = -\rho/\epsilon_0 \Rightarrow \rho = 0 \text{ except perhaps at a surface}$$

$$\nabla^2 \cancel{\vec{A}} - \mu_0\epsilon_0 \cancel{\frac{\partial^2 \vec{A}}{\partial t^2}} - \vec{\nabla}\left(\cancel{\vec{\nabla}\cdot\vec{A}} + \mu_0\epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}$$

$$\Rightarrow -\cancel{\mu_0}\epsilon_0 \vec{\nabla}\left(-\frac{kz}{\tau}e^{-t/\tau}\right) = -\cancel{\mu_0}\vec{J} \Rightarrow \boxed{\vec{J} = -\frac{k}{\tau}e^{-t/\tau}\hat{z}}$$



Two surfaces which are discharging. Current flows from one to the other uniformly.

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Use $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$

$$0 + ke^{-t/\tau}\hat{z} = \frac{\sigma}{\epsilon_0}\hat{z}$$

$$\Rightarrow \sigma = ke^{-t/\tau}\cdot\epsilon_0$$

Question 6

The vector and scalar potential in a region of space are:

$$V = V_0 \cos(kx - \omega t)$$
$$\mathbf{A} = 0$$

where $k > 0$ and $\omega > 0$.

a) Describe whether these are in either of the Coulomb or the Lorentz gauge.

Coulomb $\vec{\nabla}\cdot\vec{A} = 0$? Here $\vec{\nabla}\cdot\vec{A} = 0$ $\Rightarrow$ In Coulomb gauge

Lorentz $\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$ Here $-\frac{1}{c^2}\frac{\partial V}{\partial t} = V_0\frac{\omega}{c^2}\sin(kx-\omega t) \Rightarrow$ Not in Lorentz

b) Determine gauge transformation (via the gauge function, λ) that results in a zero scalar potential. Determine the resulting vector potential. Are these potentials in either the Coulomb or the Lorentz gauge?

Gauge transform via λ

$V \to V' = V - \frac{\partial \lambda}{\partial t}$ $\longrightarrow$ For $V' = 0$ $\frac{\partial \lambda}{\partial t} = V = V_0\cos(kx-\omega t)$

$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\lambda$

$\Rightarrow \lambda = -\frac{V_0}{\omega}\sin(kx-\omega t)$

Then $\vec{A}' = 0 + \vec{\nabla}\left(-\frac{V_0}{\omega}\sin(kx-\omega t)\right)$

$$\vec{A}' = -\frac{V_0}{\omega}k\cos(kx-\omega t)\,\hat{x} \qquad V' = 0$$

Then $\vec{\nabla}\cdot\vec{A}' = \frac{V_0 k^2}{\omega}\sin(kx-\omega t) \neq 0$ $\Rightarrow$ Not Coulomb gauge

$\frac{\partial V'}{\partial t} = 0$ $\Rightarrow$ $\vec{\nabla}\cdot A' \neq -\frac{1}{c^2}\frac{\partial V'}{\partial t}$ $\Rightarrow$ Not Lorentz gauge

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